Regular Coalition Formation*

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Abstract

In a class of stochastic games of bargaining and coalition formation with finite state space and continuous agreement sets it is shown that in an open, full measure set of parameters there exists a finite, odd number of equilibria (after removing inconsequential multiplicities of equilibrium) and each equilibrium is locally expressible as a smooth function of model parameters. The result implies existence of equilibrium and extends to certain legislative environments.

1 Introduction

I study a dynamic bargaining game of coalition formation in which a group of players may form a coalition and implement an agreement in each of a countable infinity of periods. In each period the game starts with one of a finite number of states that determine a proposer (an individual or party with the prerogative to build a coalition in that period), a set of coalitions, and feasible agreements for each of these coalitions. The set of feasible agreements may be continuous and varies with coalitions and the state. The proposer selects among proposals that take the form of a coalition and a feasible agreement, and if members of the proposed coalition unanimously approve the selected proposal, that coalition forms implementing the proposed agreement. Otherwise, a status quo agreement (that varies by state) is implemented. The game then moves to the next period with a new state according to some probability distribution that depends on the coalition that formed.

This model constitutes a generalization of sequential bargaining games originating in Rubinstein (1982) in several directions. Notably, I relax typical convexity assumptions on payoffs and feasible agreement sets, and allow bargaining to continue following agreements. I study the structure of the equilibrium set of this model and establish conditions for determinacy of equilibrium outcome distributions. After removing inconsequential multiplicities of equilibria, equilibria are characterized as solutions to a system of equations, a transformation that gives rise to a notion of regular equilibrium, in analogy to regular competitive equilibria for economies (Debreu (1970)), and regular

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Nash equilibria for games in strategic form (Harsanyi (1973)). The notion of regularity employed is strong permitting application of the classic Implicit Function Theorem at solutions of equilibrium equations. The main result is that outside a closed, measure zero set of parameters this model admits an odd number of equilibria, and each equilibrium is locally expressible as continuously differentiable function of parameters, thus satisfying a strong form of continuity. The result extends to determinacy of outcome distributions in a number of legislative settings. A byproduct of these arguments is a novel proof of existence of equilibrium.

I pursue these questions spearheaded by the development of political economy applications where such models have a natural domain, but also motivated by broader epistemological considerations shared in the literature on the structure of solution sets in games or competitive economies. I do not repeat the standard arguments here, but suffice it to say that determinacy of equilibria or outcome distributions provides needed discipline when contrasting model predictions with observables, certainly if the goal is to empirically recover model parameters. More specifically, but in a similar vein, the present study provides some redress for a common criticism leveled at non-cooperative bargaining literature for the sensitivity of its predictions on the bargaining protocol assumed. When the space of protocols is parameterized by the probabilities with which players are selected to make proposals, as is common in the political economy literature following the lead of Binmore (1987), it is established that equilibrium outcome distributions almost always change continuously (smoothly) with these protocol parameters. Such a result also appears essential in the study of general models of politics that combine voting and elections followed by bargaining for the formation of a policy or government coalition in the legislature, if the objective is to properly account for the influence of small changes in voting outcomes (through unilateral voter deviations) on subsequent coalition/policy agreements. Finally, the ability to compute equilibria becomes indispensible as the political economy literature moves from the development of tractable models that highlight intuition to more ambitious applications that aim to accommodate the data, and the present study contributes in this direction both by providing necessary theoretical foundations and formulating the building blocks to implement efficient numerical methods.

Results on the oddness of the number of equilibria in normal form games originate in Wilson (1971) and Harsanyi (1973) and have been further refined by Gul, Pearce and Stacchetti (1993). Dierker (1972) established a counterpart for competitive economies. When the model analyzed is viewed as a stochastic game, the main result of the present study is related to theorems on the determinacy of equilibria in the class of finite stochastic games (Doraszelski and Escobar (2010); Govindan and Wilson (2009); Haller and Lagunoff (2000, 2010); Herings and Peeters (2004)). With the exception of Haller and Lagunoff (2010), who study a special class of asynchronous choice repeated games, these studies feature play of a finite game in strategic form at each state, and generic determinacy of equilibrium is obtained in a product space of stage payoffs across states, with distinct payoffs assigned to each outcome (i.e., profile of actions) of the stage games. Of course, if the mapping from strategy (or action) profiles to payoff-relevant outcomes is not injective, then the relevant payoff space is null in the space of payoffs of the corresponding strategic form, and special arguments are generally needed in order to deduce determinacy or regularity of solutions. In analogous theorems for finite extensive form games, for example, an outcome corresponds to a terminal node of the game, instead of a strategy profile, and determinacy of equilibrium outcome distributions (not equilibrium behavior) is established generically in the space of outcome (i.e., terminal node) payoffs (Govindan and Wilson (2001, 2002); Kreps and Wilson (1982)). In fact,

		j			
		a_2a_3	a_2r_3	r_2a_3	r_2r_3
	x_1	x_1	x_1	x_1	x_1
i	x_2	x_2	x_2	x_1	x_1
	x_3	x_3	x_1	x_3	x_1

		i			
		a_2a_3	a_2r_3	r_2a_3	r_2r_3
	x_1	x_1	x_1	x_1	x_1
j	x_2	x_2'	x_2'	x_1	x_1
	x_3	x_3'	x_1	x_3'	x_1

Figure 1: Outcomes of strategic form at states s, s'.

Entries x_2 and x_3 in the first table indicate the implemented agreement if coalition $\{i, j\}$ forms, and x_1 the status quo agreement implemented if no coalition forms. The same outcomes can be reached at state s'; x_2' and x_3' indicate that payoffs from agreement/coalition pairs $\{i, j\}, x_c, c = 2, 3$ may differ across states.

Govindan and McLennan (2001) show that determinacy of outcome distributions need not be a generic property even in finite games in strategic form, when the payoff space is restricted by some function mapping strategy profiles to outcomes.

While the class of models I analyze constitutes a stochastic game with continuous action sets for which known determinacy results do not apply, the above discussion is relevant even in the special case (admitted by the present setup) when there is a finite number of possible agreements at each state. To be concrete, suppose that at state s there are only two active players, a proposer i and a potential coalition partner j, that the proposer can offer one of $C-1$ possible agreements $x_2, \dots, x_C$ for coalition $\{i, j\}$ to form, or unilaterally implement the status quo x_1, and that following proposal for coalition $\{i, j\}$ to form with agreement $x_c, c = 2, \dots, C$, coalition partner j decides to either approve the proposal (action a_c) or reject it (action r_c), leading to status quo agreement x_1. The first matrix in Figure 1 displays the agreements (omitting the coalition) reached in this stage game as a function of strategy (action) profiles in the case $C = 3$, and it is evident that if payoffs only depend on agreements, the space of payoffs in the stage game is significantly restricted (in the specific case with $C = 3$, from $\mathbb{R}^{24}$ to $\mathbb{R}^6$, and more generally from $\mathbb{R}^{C2^C}$ to $\mathbb{R}^{2C}$). These restrictions have bite even for the standard parameterization of the extensive form, since there are C terminal nodes (corresponding to paths of play $x_1, (x_2, r_2), \dots, (x_C, r_C)$) that lead to the same (status quo) agreement outcome.[1]

In the present analysis, an outcome of the stage game interaction is defined as an agreement/coalition pair. As a result, in the special case when the set of possible outcomes thus defined is finite, a stronger determinacy result obtains in the outcome-restricted, lower-dimensional parameter space, significantly enhancing the applicability of the main finding. A stronger form of this restriction, also commonly imposed in applications, is that the same agreements reached at two distinct states s, s', also yield the same stage payoffs. Unfortunately, generic determinacy of outcome distributions does not obtain under this stronger restriction as shown by Kalandrakis (2014) in the context of a finite bargaining game with endogenous status quo. As a result, the space of payoffs assumed in this study allows otherwise identical agreements to lead to distinct stage payoffs across

[1] This problem is not resolved if, instead, we construe each decision node of this extensive form interaction as a separate state. Then, by assigning different payoff to actions of the proposer at the (finer) state corresponding to her decision node, we would essentially assume she (and possibly other players) receive intrinsic value from her choice of different proposals.

distinct *states*. This is standard in the corresponding results for general stochastic games. Returning to the example displayed in Figure 1, if coalition $\{i, j\}$ with policies x_2 or x_3 is also possible at state s' with the same status quo x_1, but with the roles of proposer and coalition partner reversed (hence $s' \neq s$), the corresponding outcomes when a proposal passes are distinct (leading to possibly different stage payoffs, as indicated by the primes on these outcomes in Figure 1).[2] Nevertheless, the stronger form of the restriction is maintained when it comes to status quo agreements as is also indicated in Figure 1. In addition, payoffs for the same coalition/agreement pair across states differ only by a single parameter (instead of a number of parameters equal to the number of players). Consequently, the resulting parameterization has a strong claim of being the coarsest possible in this setting, in view of the indeterminacy result in Kalandrakis (2014).

The regularity result established in this study is even stronger in the general case when the space of feasible policies is continuous, as payoffs for all outcomes within each continuous feasible set vary by the same set of parameters as in the finite setting, that is, as if this feasible set were a singleton. This parametrization preserves needed structure on proposer's optimization program that allows the expression of any equilibrium as a solution to a finite dimensional system of equations, a necessary step in order to define regular equilibria. But continuous agreement sets bring fresh challenges, as they introduce new singularities in equilibrium equations compared to the standard menu of possible singularities in finite games. In particular, optimal proposals solve a nonlinear program, and may not change smoothly with parameters either because the linear independence constraint qualification (LICQ) fails at binding (exogenous and endogenous) constraints, or a subset of these constraints may bind without satisfying strict complementary slackness. This poses an aggravated instance of the analogous problem that arises in the "extended" approach to the analysis of competitive equilibria when the consumers' optimization is explicitly solved as part of the definition of equilibrium, and the resulting demand function is not assumed smooth *a priori* (e.g., Cass, Siconolfi and Villanacci (2001); Rader (1973); Shannon (1994); Smale (1974); Villanacci et al. (2002)). The problem is harder in the bargaining setting, because the binding level sets of a subset of the constraints (those corresponding to coalition partners' acceptance condition) are determined endogenously by expectation of future play so that LICQ or strict complementary slackness cannot be assumed *a priori*. In fact, as is illustrated in the motivating example of section 2, equilibrium is consistent with the coalition partners' endogenously determined constraints reducing the proposer's set of feasible proposals to a singleton, so that LICQ and any standard constraint qualification fail. Yet, such singleton acceptance sets are necessary for non-degenerate mixing on the part of coalition partners and, in turn, such voter mixing is necessary for existence of equilibrium. Fortunately, such singularities of the proposer's optimization problem are actually consistent with regularity of equilibrium, exactly because adjustments in voter and proposer mixing induce these apparently knife-edge situations to respond smoothly to small changes in parameters.

The proof of the main result relies on, standard, applications of Transversality and Sard's Theorem, and on homotopy invariance of the (modulo two) degree of equilibrium equations. The coarse parameter space along with the nature of singularities induced by the proposer's optimization program require some special arguments. In particular, in order to ensure a necessary affine independence property of proposer and binding constraint gradients at optimal policies, I develop

[2]Empirical scholars of government formation in parliamentary systems would find this a natural assumption, as it is standard to distinguish governments with the same party or even cabinet composition led by different Prime ministers (even different Prime Ministers from the same party).

an inductive argument that sequentially eliminates singularities (and the closed, measure zero sets of parameters that exhibit them) using a sequence of pseudo-equilibrium equations. This inductive approach allows the use of the same parameter at different steps of the induction to ensure a rank condition across different equilibrium equations where such singularities might emerge. A nested sequence of domains is adjusted at each step of the induction, first to ensure conditions for application of Sard's Theorem using the equilibrium manifold approach pioneered by Yves Balasko (e.g., Balasko (1988)), and subsequently to ensure properness of the mapping with a restricted domain. A standard application of the Transversality Theorem follows at termination of the induction, when the domain of the (actual) equilibrium equations resumes a product structure.

An additional complication arises in order to invoke homotopy invariance arguments to establish oddness of the number of equilibria, as this requires a set of global equilibrium equations. Since the game admits delay, there are coalitions for which no feasible agreement is acceptable so that no solution may exist for the subset of equations that characterize optimal policies for these coalitions, rendering the global system of equation infeasible even when equilibria exist. This problem is resolved by maintaining a dummy optimal proposal when no agreement is acceptable by the coalition. This dummy proposal is the solution to optimality conditions in a suitably relaxed proposer optimization problem. I ensure that positive probability is assigned to putative optimal proposals only when they are actually feasible, by building the right complementarity into equilibrium equations. The resulting homotopy constitutes a strong contender for the application of numerical homotopy continuation methods to compute equilibria of these models.

I conclude by briefly discussing additional related literature. As already mentioned, the model in this study continues in the tradition of sequential bargaining games of, e.g., Banks and Duggan (2000, 2006); Baron and Ferejohn (1989); Binmore (1987); Merlo and Wilson (1995); Rubinstein (1982), but also of models that allow for bargaining to continue and for early agreements to influence future bargaining opportunities, as in the growing literature on bargaining with endogenous status quo starting with Baron (1996). Unlike many such contributions that assume an infinite state space, the present model maintains a finite state space thus circumventing difficult issues of equilibrium existence (e.g., Duggan and Kalandrakis (2012)) and casts the genericity analysis in a finite dimensional environment. The model is closely related to the very general model analyzed by Duggan (2011), who establishes existence of equilibrium. Some of the added generality in his framework precludes the equilibrium characterization and determinacy result of this study. The two models may not be nested in variants of the legislative setting in the present analysis, when the proposer's payoff may depend on the exact voting outcome and not just on passage of the proposal. Lastly, the emphasis on the formation of coalitions, along with the agreeements these coalitions implement, connects the proposed model with the literature on coalition formation with its cooperative (e.g., Acemoglu, Egorov and Sonin (2008); Chwe (1994); Konishi and Ray (2003)) and non-cooperative incarnations (e.g., Acemoglu, Egorov and Sonin (2008); Gomes and Jehiel (2005); Hyndman and Ray (2007); Okada (1996, 2011); Ray and Vohra (1999)).

To my knowledge, the literature on the structure of the solution set of games with continuous action sets is limited. Dubey (1986) is an exception, studying generic inefficiency of Nash equilibria in strategic form games. In the bargaining literature Eraslan and McLennan (2013) use homotopy arguments to establish uniqueness of payoffs in the "divide-the-dollar" environment (see also Eraslan (2002)). The closest work to the present study is that by Kalandrakis (2006). That study focuses on a less general model and establishes a similar result for a restricted class of games that admit pure strategy equilibria. The extension of the line of argument in Kalandrakis (2006) to the set of mixed strategy equilibria (where mixing only arises on the part of proposers as the model analyzed therein does not admit delay) has proved elusive in the space of model parameters used therein. The innovation that permits the analysis in this study is to focus on coalitional voting and introduce coalition-specific payoffs that enter with the right amount of richness into equilibrium equations. Thus, these payoffs enable the wider applicability of these models for both theoretical and substantive reasons.

2 An Example

In this section I introduce a simple example (in some respects the simplest possible) of the general model studied in order to highlight some of the key issues that emerge in such environments that were discussed in the introduction and foreshadow arguments in the analysis to follow. Suppose there are three players indexed by $j = 1, 2, 3$, an agreement space $X = [-3, 3] \times [-1, 2]$, and a collection of possible coalitions $J_c \subseteq \{1, 2, 3\}$, $c = 1, \ldots, 4$, specifically, a null coalition $J_1 = \emptyset$, and coalitions $J_2 = \{1, 2\}$, $J_3 = \{1, 3\}$, and $J_4 = \{1, 2, 3\}$. Each period before one of the non-null coalitions forms, player 1 makes a proposal that consists of a coalition $J_c, c = 2, 3, 4$, and a policy $x \in X$. In keeping with later notation, denote the proposal by $(x, c) \in X \times \{2, 3, 4\}$. If all members of coalition J_c approve proposal (x, c), then coalition J_c forms, implements policy x in all subsequent periods, and the game ends. If 1's proposal is rejected, the null coalition J_1 'forms,' a status quo policy $x_\emptyset = (0, \frac{1}{2})$ is implemented in that period, and the game moves to the next period with a new round of proposing and voting. Player j's payoff from agreement (x, c) is given by a function $f_j(x, c) = -(x - \hat{x}_j)^T \cdot (x - \hat{x}_j) + u_{c,j}$, i.e., players have the typical negative quadratic preferences over policies with ideal points $\hat{x}_1 = (0, 1), \hat{x}_2 = (2, 0)$, and $\hat{x}_3 = (-2, 0)$. In addition, players receive a coalition-specific payoff $u_{c,j}$, as in Table 1. This payoff reflects consequences of the agreement (perhaps electoral consequences in a parliamentary setting?) not directly captured by the policy implemented. Overall payoff is given by the discounted sum of per period payoffs

$J_c \setminus j$	1	2	3
J_1	3	3	3
J_2	8	0	5
J_3	8	5	0
J_4	10	2	2

Table 1: Coalition-specific payoff $u_{c,j}$.

with a common discount factor $\delta = \frac{9}{10}$.

Observe that if $u_{c,j}$ is constant in c, then this example can be subsumed in the legislative environment of Banks and Duggan (2006) and, in the absence of a core policy, delay is not possible in stationary equilibria of the latter model. But with payoff depending on the coalition that forms, the agreement space violates a necessary convexity property for that no-delay result. Indeed, Table 2 details exact calculations for a stationary equilibrium with delay, specifically the policies that are proposed for each possible coalition, the probability each resulting agreement is proposed with by player 1, and the probability with which these proposals are approved by coalition partners.

Figure 2 provides a graphical summary of the equilibrium agreements and, in addition, highlights the set of acceptable policies. As already mentioned in the introduction, proposer 1 has a trivial search over optimal policies if she wants to built coalition $\{1,2,3\}$, since there is a unique policy that is acceptable by this coalition. In fact, in the equilibrium detailed in this example, this agreement is proposed with positive probability and is rejected with positive probability. Furthermore, there is a continuum of equilibrium voting strategies by the two indifferent coalition partners (i.e., players 2 and 3) after coalition $\{1,2,3\}$ is proposed with policy $x_{\{1,2,3\}}$, but all engender the same distribution over outcomes, that is, the collective acceptance probability consistent with equilibrium is uniquely pinned down in the equilibrium of Table 2.

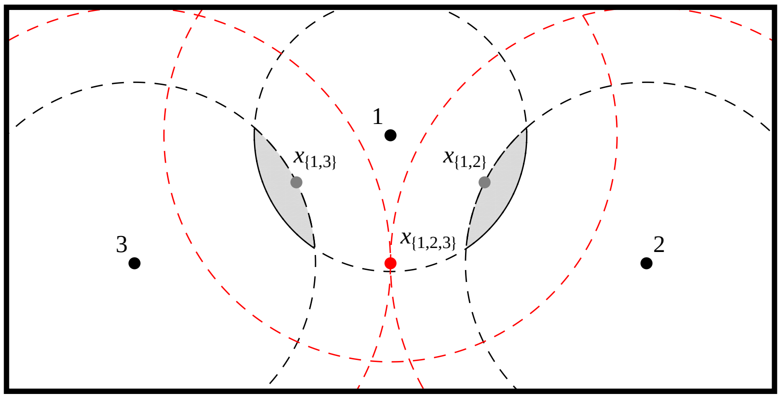

Figure 2: Acceptable policies and optimal coalition agreements in example of section 2.

The areas shaded gray correspond to policies acceptable by coalitions $\{1,2\}$ and $\{1,3\}$, with indifference contours for the respective coalition members displayed by black dashed curves. There is a unique policy (weakly) acceptable by coalition $\{1,2,3\}$ that corresponds to the red point in the intersection of the disks delineated by the red dashed indifference contours ($x_{\{1,2,3\}}$ is also the optimal policy proposal for that coalition). Ideal points of $1,2,3$ are displayed with black points, optimal policy proposals for coalitions $\{1,j\}, j=2,3$ correspond to gray points.

Coalition	Policy proposal, x_{J_c}	Proposal prob.	Collective acceptance prob.	Outcome prob.($\approx$)
J_1	$(0,\frac{1}{2})$	0	1	0.637
J_2	$(\frac{10-2\sqrt{10}}{5},\sqrt{\frac{2}{5}})$	$\frac{55+16\sqrt{10}}{1116}$	1	0.095
J_3	$(\frac{2\sqrt{10}-10}{5},\sqrt{\frac{2}{5}})$	$\frac{55+16\sqrt{10}}{1116}$	1	0.095
J_4	$(0,0)$	$\frac{503-16\sqrt{10}}{558}$	$\frac{3569+2000\sqrt{10}}{46129}$	0.174

Table 2: Equilibrium with delay and singleton collective acceptance set.

To summarize, the equilibrium of the example presented in this section features a number of properties shared by equilibria of the general model:

- There is (at most) one optimal policy proposal within the set of feasible policies for each of the three coalitions.
- Both proposer and coalition partners may employ non-degenerate mixed strategies.

- When a continuum of policies are acceptable by a coalition and preferred over status quo by the proposer, proposals to that coalition are accepted with probability one (i.e., $x_{\{1,2\}}, x_{\{1,3\}}$).
- Optimal proposals may be offered and rejected with positive probability when there is a unique acceptable policy by a coalition. In such cases, the proposer's optimization program violates standard necessary conditions for smooth dependence of solutions to parameters.
- Inconsequential indeterminacy of equilibrium behavior arises at the voting stage in the latter cases, as multiple coalition partners may be indifferent at the unique acceptable proposal.

Keeping these remarks in mind, I now move to the presentation of the model.

3 Setup

Players $1, \ldots, J$, $J \geq 2$, bargain to form a coalition and implement an agreement in each of a countable infinity of periods. Let $J_1, \ldots, J_c, \ldots, J_C$ be a collection of coalitions such that $J_c \subseteq \{1, \ldots, J\}$ for all $c = 1, \ldots, C$. The outcome of interaction in any period is represented by a pair $(x, c) \in \mathbb{R}^D \times \{1, \ldots, C\}$, interpreted as an agreement x implemented by coalition J_c that formed in that period. Bargaining proceeds as follows. Each period starts with one of S, $1 \leq S \leq C$, possible states, indexed by $s = 1, \ldots, S$. In order to flexibly model the dependence of possible coalitions and agreements on the state, partition the set $\{S+1, \ldots, C\}$ into S subsets C_s and let $\bar{C}_s = C_s \cup \{s\}$, so that $\{\bar{C}_s\}_{s=1}^S$ is a partition of $\{1, \ldots, C\}$. Let $X_c(\theta) \subset \mathbb{R}^D$, $c = 1, \ldots, C$, be the set of agreements that are feasible for coalition J_c, which may depend on parameters θ to be introduced later, and say that (x, c) is *feasible* at state s if $(x, c) \in \bigcup_{c' \in \bar{C}_s} X_{c'}(\theta) \times \{c'\}$. It is possible that $J_c = J_{c'}$ or $X_c(\theta) \cap X_{c'}(\theta) \neq \emptyset$ for distinct $c, c' \in \bar{C}_s$. At state s a proposer $i_s \in \{1, \ldots, J\}$, whose identity remains fixed across periods with state s, makes a feasible proposal (x, c) for coalition J_c to form and implement agreement x_c in that period. Subsequently players vote and if members of coalition J_c unanimously approve the proposal, then coalition J_c forms and implements agreement x. Otherwise, a *status quo* agreement x_s is implemented by a default coalition. It is assumed that $J_c = \emptyset$ and $X_c(\theta) = \{x_s\}$ when $c = s$, for all s, so that (x_s, s) at state s is feasible and implemented if proposed, that is, the proposer can always force the status quo outcome.[3] Following outcome (x, c) (possibly equal to the status quo (x_s, s)) each player j receives payoff $f_j(x, c, \theta)$ and future play is governed by a *transition state* $z_c \in \{1, \ldots, Z\}$, where $Z \leq C$ and it is assumed that $z_c \neq z_s$ for all s and all $c \in C_s$.[4] Accordingly, the game moves to the next period with a new state s' that is realized with probability $q(s'|z_c)$, and $\sum_{s'=1}^S q(s'|z_c) = 1$. Overall payoff is given by the discounted sum of within period payoffs, and player j's discount factor in a period when agreement (x, c) prevails is $\delta_j^{z_c} \in [0, 1)$.

I discuss some notation conventions before moving to the next segment. First, players are indexed by j and the mildly abusive i, instead of i_s, is reserved for the proposer at state s, whenever the

[3] Proposal (x_s, s) can also be interpreted as proposer i_s's option to 'pass' on the opportunity to build a coalition.

[4] If $z_c = z_s$ for some s and $c \in C_s$, the (unique by subsequent arguments) possible optimal agreement x_c for coalition J_c is independent of future play in the class of equilibria analyzed, and necessary regularity conditions can be forced independently of equilibrium analysis. Under such conditions, we can equivalently admit these cases in the present setting by adding optimal agreement x_c to the proposer's unilateral options, i.e., assuming $X_c(\theta) = \{x_c\}$ and $J_c = \emptyset$ when $z_c = z_s$, $c \in C_s$.

state is clear from the context. A generic transition state is denoted by z while z_c denotes the transition state reached after agreement (x, c) prevails. For any c, the state at which some outcome (x, c) is possible is denoted by $s_c \in \{1, \ldots, S\}$, so that $c \in \bar{C}_{s_c}$. Coalition J_c with and without the proposer at state s, $c \in C_s$, is denoted by $J_c^+ = J_c \cup \{i_s\}$ and $J_c^- = J_c \setminus \{i_s\}$, respectively. Cardinality of sets is denoted using $\#$, for example, $\#\{1, \ldots, S\} = S$. Set indexing of functions and vectors is used throughout, with scalars expanded as columns, for example, $f_{J_c}(x, c, \theta)$ corresponds to the (column) vector $(f_j(x, c, \theta))_{j \in J_c}$. Derivatives of scalar valued functions with respect to agreements x are denoted using the corresponding capital letters, so that $F_j(x, c, \theta) = \nabla_x f_j(x, c, \theta)$ is a $D \times 1$ (column) vector. Set indexing extends to these functions, this time expanding row-wise so that, for example, $F_{J_c}(x, c, \theta)$ corresponds to the $D \times \#J_c$ matrix formed from gradientcolumns $[F_j(x, c, \theta)]_{j \in J_c}$. Similarly, the Jacobian of any vector valued function $\phi : M \to N$ is a $\dim(M) \times \dim(N)$ matrix and it is denoted by $D\phi(y)$ whereas $D_{y'}\phi(y)$ indicates differentiation with respect to a subset of the arguments y' in the domain. Vector inequality $y \geq 0$ signifies y has non-negative entries and $y \gg 0$ means all entries of v are strictly positive.

4 Regularity assumptions

Most of the results established for the above coalition formation game hold in an open, full measure subset of an open set of parameters Θ. These arguments require certain regularity assumptions on the manner parameters $\theta \in \Theta$ enter feasible policy sets and payoffs which are introduced in this section. First, it is assumed that payoff functions $f_j : \mathbb{R}^D \times \{1, \ldots, C\} \times \Theta \to \mathbb{R}$ satisfy $f_j(x, c, \theta) > 0$ for all j, all $\theta \in \Theta$, and all feasible agreements (x, c), a restriction that does not entail any loss of generality. Feasible policy sets $X_c(\theta)$ are defined by some combination of K inequality constraints $g_k : \mathbb{R}^D \times \Theta \to \mathbb{R}$, indexed by $k = J+1, \ldots, J+K$, and L equality constraints $h_\ell : \mathbb{R}^D \to \mathbb{R}$, indexed by $\ell = J+K+1, \ldots, J+K+L$. In particular,[5]

$$X_c(\theta) = \{x \in \mathbb{R}^D \mid g_{K_c}(x, \theta) \geq 0, h_{L_c}(x) = 0\},$$

for all c, where $K_c \subseteq \{J+1, \ldots, J+K\}$, $L_c \subseteq \{J+K+1, \ldots, J+K+L\}$. As is standard, agreement sets, $X_c(\theta)$, are assumed non-empty and bounded.

(A1) Boundedness: *For all c there exists compact X_c such that $\emptyset \neq X_c(\theta) \subseteq X_c$ for all θ.*

For fixed $\bar{\varepsilon} > 0$, and for all c define the open set

$$Y_c = \cup_{x \in X_c} \{y \mid ||x - y|| < \bar{\varepsilon}\},$$

on which payoff and constraint functions are assumed suitably smooth:

(A2) Smoothness: *f_j, g_k, and h_ℓ are C^2 on $Y_c \times \Theta$ for all c and all j, k, ℓ.*

[5]This formulation subsumes the case of feasible sets $X_c(\theta), c = 1, \ldots, S$, that contain the status quo policies x_s, by setting $K_s = \emptyset$ and assuming $L_s = \{\ell_{s,1}, \ldots, \ell_{s,D}\} \subseteq \{J+K+1, \ldots, J+K+L\}$, where $h_{\ell_{s,d}}(x) = x_d - x_{s,d}$, $d = 1, \ldots, D$.

As is the case for payoff functions, gradients of the constraint functions with respect to x are denoted by $G_k(x,\theta) = \nabla_x g_k(x,\theta)$ and $H_\ell(x) = \nabla_x h_\ell(x)$. Note that **(A1)**-**(A2)** ensure, among other things, compactness of $X_c(\theta)$ for all possible parameters θ.

I now decompose the parameter vector θ into $(C-S)$ scalars θ_c, Z vectors $\theta^z \in \mathbb{R}^J$, and a vector $\theta^0 \in \mathbb{R}^K$, so that $\theta = (\theta^0, (\theta^z)_{z=1}^Z, (\theta_c)_{c=S+1}^C) \in \Theta \subset \mathbb{R}^{K+Z\cdot J+C-S}$, and assume:

(A3) Parameters: *For all c, all $x \in Y_c$, all θ, and all j,k, $g_k(x,\theta) = g_k(x) + \theta_k^0$ and*

$$f_j(x,c,\theta) = \begin{cases} f_j(x,c,\theta_c) + \theta_j^{z_c} & \text{if } c > S, \\ f_j(x,c) + \theta_j^{z_c} & \text{if } c \leq S, \end{cases} \quad \text{and} \quad \frac{\partial f_j(x,c,\theta_c)}{\partial \theta_c} > 0, \text{ if } c > S, j \in J_c^+.$$

Payoff parameter θ_c is a coalition- and state-specific parameter affecting joint coalition payoffs from agreement (x,c). On the other hand, payoff parameters θ_z are specific to the transition state reached by any coalition agreement and vary by player. One possible interpretation of the role of these two sets of payoff parameters is to identify transition states z and the corresponding parameters θ_z with the systematic consequences of any coalition agreement, which may be common across proposers or states with distinct coalition building opportunities (when $z_c = z_{c'}$ for distinct c, c'), and let parameter θ_c capture idiosyncratic (e.g., proposer-specific) payoff consequences of outcome (x,c) for coalition members (and possibly other players). By separating these two sets of payoff parameters we obtain generic properties of equilibrium in a potentially lower-dimensional set of parameters.[6] Note that θ^z is restricted to enter payoffs in a linear additive fashion, a restriction of no consequence when $X_c(\theta)$ is a singleton that also preserves needed structure on the proposer's optimization program when $X_c(\theta)$ is a continuum. On the other hand, the effect of parameter θ_c may depend quite generally on the agreement x. Finally, parameter θ_k^0 modifies the binding level of the corresponding inequality constraint, g_k, ensuring such constraints can only bind with strict complementary slackness at any optimal proposal.

With this parameterization of the coalition formation game fixed, I now introduce three additional assumptions that impose structure on proposers' optimization over policies. Although some of the properties introduced with these assumptions may also be obtained generically using existing parameters or by enriching the space of model parameters, directly assuming these properties allows focus on the core issues that arise in establishing regularity of these coalition formation games. In that vein, the gradients of equality and binding inequality constraints are assumed linearly independent. To state that assumption, first define the set of binding inequality constraints at agreement x, c and θ by

$$K_c(x,\theta) = \{k \in K_c \mid g_k(x,\theta) = 0\}.$$

(A4) LIC: *For all c, all θ, and all $x \in Y_c$, $[G_{K_c(x,\theta)}(x,\theta)\ H_{L_c}(x)]$ has full column rank.*

While it may appear overtly restrictive to require that **(A4)** hold for all θ, note that linear independence is an open property so that if it holds at some θ then it holds in an open set containing that

[6] In particular, the main conclusions of the analysis hold *a fortiori* if we replace parameters $((\theta_c)_c, (\theta_j^z)_{z,j}) \in \mathbb{R}^{C-S+Z\cdot J}$ with parameters $(\theta_j^c)_{c,j} \in \mathbb{R}^{C\cdot J}$. Of course, any advantage of the maintained parameterization disappears if the mapping from $\{1,\ldots,C\}$ to $\{1,\ldots,Z\}$ is injective.

point. Moreover, **(A4)** far from implies the *linear independence constraint qualification* (LICQ) for the proposer's optimization over policy agreements, since it does not address acceptance constraints imposed by coalition partners.

Next, I introduce a key joint shape restriction on payoff and constraint functions that constitutes a generalization of convexity. Let $J_c(x, x', \theta) = \{j \in J_c^+ \mid f_j(x', c, \theta) \geq f_j(x, c, \theta)\}$ denote the set of members of coalition J_c (and the proposer) who weakly prefer coalition agreement (x', c) over alternative (x, c).

(A5) P-Invexity: *For all c, all θ, and all distinct $x, x' \in X_c(\theta)$ there exists λ such that*

$$\lambda^T \cdot F_{J_c(x,x',\theta)}(x, c, \theta) > 0, \lambda^T \cdot G_{K_c(x,\theta)}(x, \theta) \geq 0, \text{ and } \lambda^T \cdot H_{L_c}(x) = 0.$$

(A5) requires that for any agreement x and any sub-coalition that weakly prefers x' over x, there exists a direction of strict improvement from x for all members of the sub-coalition and is consistent with equality and binding inequality constraints. As shown in Lemma 1, **(A5)** renders a version of the Fritz-John optimality conditions necessary and sufficient for proposal (x, c) to solve $P(c; \sigma)$ and guarantees uniqueness[7] of optimal policies in $X_c(\theta)$. By ruling out the potential multiplicity of optimal policies we can express all equilibria as solutions to a finite- (and fixed-) dimensional system of equations. **(A5)** is related to variants termed Pseudo-invexity in the literature on multi-objective programming (e.g., R. Osuna-Gomez (1998), Arana-Jimenez et al. (2008)). The essence of these conditions is that they retain properties of convex optimization problems without actually requiring concavity of the objective and constraint functions. Standard weakenings of concavity (e.g., h_ℓ is linear, g_k and f_j pseudo-concave (Diewert, Avriel and Zang (1981)) in Y_c) along with a *limited shared weak preference* assumption (Banks and Duggan (2000)) imply **(A5)**.[8]

Lastly, a version of the familiar Strong Second Order sufficient Condition is assumed.

(A6) SSOC: *For all c, all θ, and all $x \in Y_c$, if there exist $\hat{J} \subseteq J_c^+$, $\hat{K} \subseteq K_c(x, \theta)$, and $\beta_f \gg 0, \beta_g \gg 0, \beta_h$ such that $F_{\hat{J}}(x, c, \theta) \cdot \beta_f + G_{\hat{K}}(x, \theta) \cdot \beta_g + H_{L_c}(x) \cdot \beta_h = 0$, then*

$$\lambda^T \cdot D_x(F_{\hat{J}}(x, c, \theta) \cdot \beta_f + G_{\hat{K}}(x, \theta) \cdot \beta_g + H_{L_c}(x) \cdot \beta_h) \cdot \lambda < 0,$$

for all $\lambda \in \mathbb{R}^D$ such that $\lambda^T \cdot F_{\hat{J}}(x, c, \theta) = 0$, $\lambda^T \cdot G_{\hat{K}}(x, \theta) = 0$, and $\lambda^T \cdot H_{L_c}(x) = 0$.

The equation stated right before the displayed inequality in **(A6)** constitutes a version of the familiar Fritz-John necessary conditions for optimality, which characterize optimal proposals by virtue of **(A5)**. Note that a related second order *necessary* condition requires that the weak version of the displayed inequality holds for a similar set of directions λ,[9] and the gap between

[7] But not uniqueness of optimal feasible agreements at state s, since the latter is chosen from the union of feasibility sets $X_c(\theta)$ for all $c \in \bar{C}_s$.

[8] E.g., a sufficient version of LSWP is: *For all c, all $\bar{f}_j \in \mathbb{R}, j \in J_c^+$, either*

$$\text{cl}(\{x \in X_c(\theta) \mid f_{J_c^+}(x) > \bar{f}_{J_c^+}\}) = \{x \in X_c(\theta) \mid f_j(x) \geq \bar{f}_j, j \in J_c^+\} \text{ or } \#\{x \in X_c(\theta) \mid f_{J_c^+}(x) \geq \bar{f}_{J_c^+}\} \leq 1.$$

[9] This necessary condition adds the requirement that λ nullify or render positive the gradient of any binding inequality constraints with zero multipliers. See, e.g., Still and Streng (1996), Theorem 3.4. The required sets of direction coincide, though, when strict complementary slackness holds, a property that is shown to hold generically.

nonlinear programs that satisfy these necessary conditions for optimality and the above sufficient condition is small (e.g., Spingarn and Rockafellar (1979)). Keeping these remarks in mind, under **(A4)**, a sufficient condition for SSOC is that all the constraints are pseudo-concave and all f_j strongly pseudo-concave in Y_c (Diewert, Avriel and Zang (1981)).

5 Equilibrium

I focus the analysis on stationary subgame perfect equilibria in Markov strategies. A proposal strategy for i at state s is a Borel probability measure π_s on $\mathbb{R}^D \times \bar{C}_s$. A voting strategy for j is a Borel measurable function $\alpha_j : \mathbb{R}^D \times \{1, \ldots, C\} \to [0, 1]$.[10] Accordingly, $\alpha_j(x, c)$ is the probability that j approves proposal (x, c). Denote a profile of strategies by $\sigma = ((\pi_s)_{s=1}^S, (\alpha_j)_{j=1}^J)$ and let the collective acceptance probability of proposal (x, c) be[11]

$$\alpha(x, c; \sigma) = \begin{cases} \prod_{j \in J_c} \alpha_j(x, c) & \text{if } x \in X_c(\theta) \\ 0 & \text{otherwise.} \end{cases}$$

Given strategy profile σ, the value of interim state z for player j is obtained recursively as

$$V_j(z; \sigma) = \sum_{s=1}^{S} q(s \mid z) \int_{\mathbb{R}^D \times \bar{C}_s} \Big(\alpha(x, c; \sigma)(f_j(x, c, \theta) + \delta_j^{z_c} V_j(z_c; \sigma)) \\ + (1 - \alpha(x, c; \sigma))(f_j(x_s, s, \theta) + \delta_j^{z_s} V_j(z_s; \sigma)) \Big) \, d\pi_s,$$

and the expected payoff of j from outcome (x, c) is

$$U_j(x, c; \sigma) = f_j(x, c, \theta) + \delta_j^{z_c} V_j(z_c; \sigma).$$

In order to define an equilibrium concept suitable for the arguments that follow, start with two reasonably minimal equilibrium conditions. The first condition is that proposers optimize, that is,

$$\text{(P)} \qquad \pi_s(\arg \max_{(x,c) \in \mathbb{R}^D \times \bar{C}_s} \{\alpha(x, c; \sigma) U_i(x, c; \sigma) + (1 - \alpha(x, c; \sigma)) U_i(x_s, s; \sigma)\}) = 1, \text{ for all } s.$$

The second condition similarly requires that voters optimize:

$$\text{(V)} \qquad \alpha_j(x, c) = \begin{cases} 1 & \text{if} \quad U_j(x, c; \sigma) > U_j(x_s, s; \sigma) \\ 0 & \text{if} \quad U_j(x, c; \sigma) < U_j(x_s, s; \sigma), \end{cases} \quad \text{for all } s, \text{ all } c \in C_s, \text{ all } j, \text{ and all } x.$$

Equilibrium condition (V) already reflects a standard refinement by requiring that voters behave as if pivotal, notably, coalition members approve (reject) proposals they (do not) strictly prefer over the status quo, even if some other coalition member votes to reject, in which case their vote does not influence approval of the proposal.

[10] Anticipating an extension to legislative settings, α_j specifies redundant votes on proposals (x, c) even if $j \notin J_c$.

[11] Note that $\alpha(x, c; \sigma) = 1$ if $x \in X_c(\theta)$ and $J_c = \emptyset$.

These equilibrium conditions leave room for at least three types of benign indeterminacy of equilibrium behavior that do not affect the distribution over outcomes at each state. First, if implementing the status quo agreement, (x_s, s), is optimal for the proposer at state s, the proposer may be randomizing in equilibrium among a continuum of alternatives that are rejected with probability one. Second, a proposer may be indifferent and mixing between a proposal $(x, c) \neq (x_s, s)$, that is rejected with positive probability, and the status quo (x_s, s). In such cases, an increase in the probability of proposing (x, c) at the expense of proposing (x_s, s) can be balanced by a decrease in the collective probability of accepting proposal (x, c), leaving the overall distribution over implemented agreements unaffected. A third form of indeterminacy emerges when multiple coalition partners are indifferent between a proposal (x, c) and the status quo (x_s, s), thus allowing a continuum of mixed voting strategies that induce the same collective acceptance probability $\alpha(x, c; \sigma) \in (0, 1)$, as actually occurs in the example of section 2.

The possibility of this type of benign nondeterminacy of equilibrium behavior is well known to emerge even in generic sequential move games (Govindan and McLennan (2001); Govindan and Wilson (2001, 2002); Kreps and Wilson (1982)). In the same spirit as in this literature, the results of this study pertain to determinacy of equilibrium outcome distributions, instead of determinacy of equilibrium behavior. To facilitate focus on outcome distributions, I define a mildly refined notion of equilibrium strengthening conditions (P) and (V) in a manner that limits the three sources of multiplicity of equilibrium behavior identified in the previous paragraph, without restricting the set of equilibrium outcome distributions. When it comes to voting behavior, this refinement allows focus on collective acceptance probabilities, instead of individual votes, as the former suffice to characterize outcome distributions. To develop these arguments define the maximal and minimal, respectively, set of policies that can be approved by (perhaps non-equilibrium) voter best responses to i's proposal (x, c) at state s, given strategy profile σ as

$$\bar{A}(c; \sigma) = \{x \in X_c(\theta) \mid U_j(x, c; \sigma) \geq U_j(x_s, s; \sigma), j \in J_c^-\}, \text{ and}$$
$$A(c; \sigma) = \{x \in X_c(\theta) \mid U_j(x, c; \sigma) > U_j(x_s, s; \sigma), j \in J_c^-\}.$$

Also, define the optimization problem $P(c; \sigma)$

$$P(c;\sigma) \qquad \max_x f_i(x, c, \theta) + \delta_i^{z_c} V_i(z_c; \sigma) \quad s.t.$$
$$x \in \bar{A}(c; \sigma),$$

and collect all solutions of these programs for state s in the set

$$\Pi(s; \sigma) = \left\{(x, c) : x \text{ solves } P(c; \sigma), c \in \bar{C}_s\right\}.$$

With these preliminaries, an equilibrium is defined as a two-pronged strengthening of conditions (P) and (V).

Definition 1. *Strategy profile σ is an* equilibrium *if it satisfies conditions (P) and (V), and*

1. *Proposal strategies satisfy $\pi_s(\Pi(s; \sigma)) = 1$ for all s.*

2. *For all s, all $c \in C_s$, all x, and all $j', j'' \in \{j \mid U_j(x, c; \sigma) = U_j(x_s, s; \sigma)\}$, $\alpha_{j'}(x, c) = \alpha_{j''}(x, c)$; and if $A(c; \sigma) \neq \emptyset$ then $\alpha_{j'}(x, c) = 1$.*

Accordingly, the equilibrium concept restricts strategies in two additional directions. On the one hand, a proposer can place positive probability only to candidate optimal proposals in $\Pi(s;\sigma)$. This refinement can be viewed as a perfectness condition, as it prevents the proposer from offering strictly inferior agreements she expects to be rejected. On the other hand, indifferent voters vote to accept proposals with probability one, unless the set of agreements that are strictly acceptable by the proposed coalition is empty in which case all indifferent voters use the same mixing probability. It follows that non-degenerate voter mixing can occur in equilibrium only if $A(c;\sigma)=\emptyset$.

In what follows, say that two strategy profiles are *equivalent* if they induce the same outcome distributions and payoffs. We can show that every strategy profile that satisfies the less restrictive conditions (P) and (V) can be converted to an equivalent equilibrium:

Theorem 1. *For every profile σ that satisfies (P) and (V) there exists an equivalent equilibrium profile σ'.*

In view of Theorem 1, the added restrictions of Definition 1 entail no harm in that all outcome distributions and payoffs consistent with the unrestricted equilibrium conditions also emerge in an equivalent (restricted) equilibrium.

6 Equilibrium Equations

The equilibrium concept opens the way for expressing equilibria as solutions to a finite-dimensional system of equations. A key result in that direction is that a version of the Fritz-John optimality conditions (which appear as equations (2) in the appendix) are necessary and sufficient at any solution of program $P(c;\sigma)$ and, in fact, such a solution is unique (when it exists).

Lemma 1. *For all c and all strategy profiles σ,*

1. *x solves $P(c;\sigma)$ if and only if there exists $b=(b_r)_{r\in J_c^+\cup K_c\cup L_c}$ such that x,b solve (2).*
2. *If $\bar{A}(c;\sigma)\neq\emptyset$ then there is a unique x that solves $P(c;\sigma)$.*
3. *There exist x,b, with $b_i=0$, that solve (2) if and only if $\bar{A}(c;\sigma)=\{x\}$ and $A(c;\sigma)=\emptyset$.*
4. *For all $x\in\bar{A}(c;\sigma)$, if $\bar{A}(c;\sigma)\neq\{x\}$ then there exists $\tau>0$ and a C^1 function $\gamma:[0,\tau]\to\mathbb{R}^D$ such that $\gamma(0)=x$ and $\gamma(t)\in A(c;\sigma)$ for all $t\in(0,\tau]$.*

Part 4 of Lemma 1 ensures that every weakly acceptable policy can be arbitrarily closely approximated by strictly acceptable policies, unless the set of weakly acceptable policies is a singleton, a result that underpins Theorem 1 and justifies the refinement that non-degenerate voter mixing be restricted to cases when the set of strictly acceptable agreements, $A(c;\sigma)$, is empty. Jointly, Lemma 1 and Theorem 1 imply that at any state s there can be at most $\#\bar{C}_s$ outcomes (x,c), possibly including the status quo (x_s,s), that are proposed and implemented with positive probability in any equilibrium. This recasts equilibria as finite-dimensional objects and suggests an obvious path for the formulation of a system of equations characterizing equilibria. On the one hand, we can use the necessary and sufficient conditions identified in part 1 of Lemma 1 to isolate candidate optimal agreements in $\Pi(s;\sigma)$ for all s and limit proposal strategies to such agreements. On the other hand, part 3 of Lemma 1 provides a means to identify cases when non-degenerate mixing is possible at the voting stage, by requiring a zero multiplier for the proposer in these optimality conditions. Furthermore, when $A(c;\sigma) = \emptyset$ we also conclude that $\bar{A}(c;\sigma)$ is a singleton and there must exist some $j \in J_c^-$ that is indifferent between the proposal and the status quo agreements so that non-degenerate mixing is consistent with equilibrium condition (V).

Part 2 of Lemma 1 also identifies a potential problem with this strategy: it is possible that there exist equilibria σ and c such that no proposal is weakly acceptable ($\bar{A}(c;\sigma) \neq \emptyset$) so that if optimization conditions for all c are included in equilibrium equations, such equilibria cannot be recovered because program $P(c;\sigma)$ has no solution. One resolution of this problem is to work with local equilibrium equations that isolate equilibria with support on some subset of candidate agreements (x, c). The main drawback of this approach is that, since we cannot know a priori the subset of $c > S$ that may exhibit an empty weakly acceptable set of proposals, we must separately consider all possible combinations of such sets. This is theoretically possible (there are only a finite number of such combinations) but produces a combinatorial nightmare when it comes to computing equilibria. We avoid the need for such local arguments by carefully building the right complementarity into optimization conditions (2) and construct a set of global equilibrium equations in this section. This global approach has three main advantages. First, it provides an avenue to establish existence of equilibrium. Second, it yields a stronger theorem on the determinacy of equilibria. Third, it paves the way for the efficient computation of equilibria using homotopy methods.

By virtue of Lemma 1, an equilibrium profile σ can be reduced to a vector

$$e = ((x_c, b_c, a_c)_{c=S+1}^C, (p_c)_{c=1}^C, (V^z)_{z=1}^Z),$$

where $x_c \in \mathbb{R}^D$ is a (putative) optimal policy solving $P(c;\sigma)$, $b_c \in \mathbb{R}^{\#(J_c^+ \cup K_c \cup L_c)}$ the corresponding mulitpliers, scalars a_c, p_c determine the collective acceptance probability and probability of proposal (x_c, c), respectively, and $V^z \in \mathbb{R}^J$ is the vector of continuation values of transition state z. Define the space $E = \mathbb{R}^{\sum_{c=S+1}^C (D+\#(J_c^+ \cup K_c \cup L_c)+2)+S+JZ}$ where such vectors e reside. In order to write equilibrium equations more compactly, we introduce the following three auxiliary functions. First, let $[\cdot]_+ : \mathbb{R} \to \mathbb{R}$ be any C^1 function that is strictly monotone increasing in $\mathbb{R}_{++}$ and identically zero in $(-\infty, 0]$. Similarly, let $[\cdot]_0^1 : \mathbb{R} \to \mathbb{R}$ be any C^1 function that is strictly monotonic in $(0, 1)$, is identically zero in $(-\infty, 0]$, and identically 1 in $[1, +\infty)$.[12] Finally, let

$$U_j(e, c, \theta) = f_j(x_c, c, \theta) + \delta_j^{z_c} V_j^{z_c}$$

[12] For example, $[x]_+ = \max^2\{0, x\}$, and $[x]_0^1 = \min\{1, \max\{0, -2x^3 + 3x^2\}\}$.

be the expected payoff from putative agreement (x_c, c) if continuation payoffs are given by $(V^z)_{z=1}^Z$. Equilibrium equations now take the form:

$$F_{J_c^+}(x_c, c, \theta) \cdot [b_{c,J_c^+}]_+ + G_{K_c}(x_c, \theta) \cdot [b_{c,K_c}]_+ + H_{L_c}(x_c) \cdot b_{c,L_c} = 0 \tag{1a}$$

$$U_{J_c^-}(e, c, \theta) - U_{J_c^-}(e, s_c, \theta) - [-b_{c,J_c^-}]_+ + \mathbf{1} \cdot [-a_c]_+ = 0 \tag{1b}$$

$$g_{K_c}(x_c, \theta) - [-b_{c,K_c}]_+ = 0 \tag{1c}$$

$$h_{L_c}(x_c) = 0 \tag{1d}$$

$$\mathbf{1}^T \cdot [b_{c,J_c^+}]_+ - 1 = 0 \tag{1e}$$

$$[-p_{s_c}]_+ - [a_c]_0^1(U_i(e, c, \theta) - U_i(e, s_c, \theta)) - [-p_c]_+ + [-a_c]_+ = 0, \tag{1f}$$

$$b_{c,i} - [a_c - 1]_+ = 0 \tag{1g}$$

for all $c = S + 1, \ldots, C$,

$$\sum_{c \in \bar{C}_s} [p_c]_+ - 1 = 0, \tag{1h}$$

for all s, and

$$V_j^z - \sum_{s=1}^{S} q(s|z) \sum_{c \in \bar{C}_s} [p_c]_+([a_c]_0^1(U_j(e, c, \theta) + (1 - [a_c]_0^1)U_j(e, s_c, \theta)) = 0, \tag{1i}$$

for all z, j.

A few comments are in order. First, for convenience, I have made use of the identity $a_c = 1$ for all $c \leq S$, and these variables do not actually appear in e. Second, equations (1a)-(1e) are essentially the Fritz-John conditions (2) of Lemma 1, except for the additional dependence on scalar $a_c, c > S$, which plays a triple role. The value of a_c in excess of unity mirrors the multiplier of the proposer's objective function in the Fritz-John conditions when the latter is positive; it is (a monotone function of) the probability with which the proposal (x_c, c) is accepted when it ranges over the unit interval and non-degenerate voter mixing occurs; and it indicates the degree of *infeasibility* of the nonlinear program $P(c; \sigma)$ when negative. Equations (1g) exploit part 3 of Lemma 1 in order to enforce the second part of the definition of equilibria, that is, they ensure that agreement (x_c, c) is collectively accepted with probability one if the proposer's multiplier $b_{c,i}$ is strictly positive reserving non-degenerate mixing only for cases when the set of strictly acceptable policies in $X_c(\theta)$ is empty. Equations (1f) and (1h) jointly ensure that the proposer optimizes by proposing exclusively in set $\Pi(s; \sigma)$ at state s. In particular, (1f) and (1h) are equivalent to

$$\bar{U}_i - ([a_c]_0^1 U_i(e, c, \theta) + (1 - [a_c]_0^1)U_i(e, s, \theta)) = [-p_c]_+ - [-a_c]_+,$$

for all s and all $c \in \bar{C}_s$, where $\bar{U}_i = \sum_{c' \in \bar{C}_s} [p_{c'}]_+([a_{c'}]_0^1 U_i(e, c', \theta) + (1 - [a_{c'}]_0^1)U_i(e, s, \theta))$ is the proposer's expected payoff.[13] If (x_c, c) is infeasible, then $[-a_c]_+ > 0$ and $p_c \leq 0$. If (x_c, c) is

[13] To see the equivalence, multiply each of the above equations corresponding to c with $[p_c]_+$ and sum across $c \in \bar{C}_s$ to obtain (1h), after invoking the property that $U_i(e, c, \theta) > 0$ for all c.

feasible, then $[-a_c]_+ = 0$ and $[-p_c]_+$ measures the sub-optimality of proposal (x_c, c). In that light,

$$[-p_s]_+ = \sum_{c' \in \bar{C}_s} [p_{c'}]_+ [a_{c'}]_0^1 (U_i(e, c', \theta) - U_i(e, s, \theta)) = \bar{U}_i - U_i(e, s, \theta),$$

that is, $[-p_s]_+$ is the difference between the proposer's attained expected payoff and that guaranteed by the status quo. Finally, equations (1i) determine the value of transition states when future play is governed by such voting and proposal strategies. Collect the left-hand-side of equations (1) into a function $\Phi : E \times \Theta \to \mathbb{R}^{\dim(E)}$.

To conclude this section, define a function $e \mapsto \hat{\sigma}(e)$ that converts vectors e that solve $\Phi(e, \theta) = 0$ to a strategy profile $\sigma = \hat{\sigma}(e)$. In particular, let $\hat{\sigma}(e)$ be generated by requiring that proposal strategy satisfy $\pi_s(\{(x_c, c)\}) = [p_c]_+$ for all s and all $c \in \bar{C}_s$, and

$$\alpha_j(x, c) = \begin{cases} 1 & \text{if } U_j(e,c,\theta) > U_j(e,s,\theta), \text{ or } U_j(e,c,\theta) = U_j(e,s,\theta) \text{ and } A(c,e) \neq \emptyset, \\ 0 & \text{if } U_j(e,c,\theta) < U_j(e,s,\theta), \\ \sqrt[J_I(e,c)]{[a_c]_0^1} & \text{if } U_j(e,c,\theta) = U_j(e,s,\theta), \text{ and } A(c,e) = \emptyset, \end{cases}$$

for all j, all x, all s, and all $c \in C_s$, where $J_I(e,c) = \#\{j \in J_c \mid U_j(e,c,\theta) = U_j(e,s,\theta)\}$, $\bar{A}(c,e) = \{x \in X_c(\theta) \mid U_j(e,c,\theta) \geq U_j(e,s,\theta), j \in J_c^-\}$, and $A(c,e) = \{x \in X_c(\theta) \mid U_j(e,c,\theta) > U_j(e,s,\theta), j \in J_c^-\}$. Note that $\alpha(x_c, c; \hat{\sigma}(e)) = [a_c]_0^1$ for all c and $\sum_{c \in \bar{C}_s} \pi_s(\{(x_c, c)\}) = 1$ for all s, by equation (1h). Furthermore, $\hat{\sigma}(e)$ immediately satisfies equilibrium restrictions imposed by condition (V) and Definition 1 and it also satisfies equilibrium condition (P) by equations (1f) and (1h). Indeed, the following theorem establishes that the mapping Φ has the right properties.

Theorem 2. *For all $\theta \in \Theta$, σ is an equilibrium if and only if there exists e such that $\Phi(e, \theta) = 0$ and $\sigma = \hat{\sigma}(e)$.*

In view of Theorem 2, every equilibrium σ corresponds to some e that solves equations (1) and for every e that solves these equations strategy profile $\hat{\sigma}(e)$ is an equilibrium. Equilibrium equations $\Phi(e, \theta) = 0$ are used to study the structure of the equilibrium set in the next section.

7 Regular Equilibrium

Expressing equilibria as solutions to a system of equations makes it possible to introduce a second equilibrium concept, that of a *regular* equilibrium. Regularity is a well established concept in finite normal form games and has been extended to finite stochastic games by a number of authors. To my knowledge, there is no standard analogue for continuous action stochastic games of the type analyze in this study (though Kalandrakis (2006) is a related precursor). The definition of regularity I work with builds on the shared feature of existing definitions, that is, the property that equations characterizing equilibrium have a non-vanishing Jacobian with respect to endogenous quantities.

Definition 2. *Strategy profile σ is a* regular *equilibrium if there exists e^* such that $\sigma = \hat{\sigma}(e^*)$, $\Phi(e^*, \theta) = 0$, and $\operatorname{rank}(D_e\Phi(e^*, \theta)) = \dim(E)$.*

I motivated Definition 2 on the obvious analogy with definitions of regularity for finite games concerning the non-singularity of the Jacobian of equilibrium equations,[14] but the nature of the restrictions introduced by this refinement is so far quite vague. This is rectified in the next lemma that sheds light on key properties of regular equilibria:

Lemma 2. *If σ is a regular equilibrium then there exists a unique e^* such that $\sigma = \hat{\sigma}(e^*)$ and $\Phi(e^*, \theta) = 0$. Furthermore, e^* is such that:*

1. *$p^*_c \neq 0$ for all c.*
2. *For all $c > S$, $a^*_c \notin \{0, 1\}$ and if $a^*_c \in (0, 1)$ then $p^*_c > 0$.*
3. *$b^*_{c,r} \neq 0$ for all $c > S$ and all $r \in J^-_c \cup K_c$.*
4. *The matrix $\begin{pmatrix} F_{j \in J^+_c : b^*_{c,j} > 0}(x^*_c, c, \theta) & G_{k \in K_c : b^*_{c,k} > 0}(x^*_c, \theta) & H_{L_c}(x^*_c) \\ \mathbf{1}^T & 0 & 0 \end{pmatrix}$ has full column rank for all $c > S$.*

The first of the four implications stated in Lemma 2 amounts to a version of the quasi-strictness property of regular Nash equilibria in normal form games, namely, any unused proposer actions (i.e., agreements, in or outside $\Pi(s; \sigma)$, that are proposed with probability zero) are strictly inferior in expected utility terms compared to those actions chosen with positive probability. Similarly, the third part of the Lemma asserts that strict complementary slackness holds for all optimality conditions characterizing potential optimal policy proposals at a regular equilibrium. A constraint that binds without strict complementary slackness opens the possibility for 'kinks' in the manner optimal proposals (and equilibrium) change with parameters. The last implication goes one step further and requires the affine independence of equality, binding inequality, and (possibly) proposer gradients at optimal agreement x^*_c of program $P(c; \sigma)$. The possible exclusion of the proposer from this condition presents an unusual regularity condition for this optimization program, one that is relevant when weakly acceptable policies reduce to a singleton and proposer's multiplier is zero. In those cases, all standard constraint qualifications (certainly LICQ) fail, yet equilibrium may still be regular. Indeed, in the second part of Lemma 2 a related property is asserted in cases when $\bar{A}(c; \sigma)$ is a singleton with $A(c; \sigma) = \emptyset$. Such knife-edge cases cannot appear in a regular equilibrium σ, *unless* the proposer offers these weakly feasible proposals with positive probability and they get accepted with positive probability (but not with probability one). Intuitively, mixing on the part of the voters provides the extra degree of freedom that restores robustness to these fragile feasible agreements.

Fortunately, the long list of singularities identified by Lemma 2 can only occur in a closed, measure zero subset of parameters.

Lemma 3. *There exists an open, full measure subset of Θ, $\Theta^* \subseteq \Theta$, such that for all $\theta \in \Theta^*$ all equilibria are regular.*

[14] Working with global equilibrium equations is not innocuous in that regard as this version of regularity imposes restrictions on auxiliary quantities that are not part of actual equilibrium, such as variables that appear in equations corresponding to infeasible policy optimization programs for the proposer. This heavy-handed approach, though, is not consequential for the main conclusion of Theorem 4.

The proof of Lemma 3 relies on a delicate induction argument executed in Lemmas 4 to 7 in the Appendix. In these Lemmas, a sequence of pseudo-equilibrium equations isolate and sequentially eliminate a closed set of measure zero of parameters that induce a subset of the singularities identified by Lemma 2 at each step. At termination of this inductive argument, the pseudo-equilibrium equations reduce to equilibrium equations (1). In order for the results obtained on preceding pseudo-equilibrium equations to apply at each inductive step, the domains of successive pseudo-equilibrium mappings are nested. Typically, it is possible to remove such singularities in separate independent steps, for example, when equilibria that are not quasi-strict are ruled out after establishing that (restricted) totally mixed equilibria are generically regular. Such an approach is not possible in the present setting because the coarse parameter space does not leave enough variables to simultaneously ensure the necessary rank condition for the Jacobian of (pseudo-)equilibrium conditions across states and coalitions. The inductive approach makes efficient use of the austere parameterization assumed in **(A3)** and is designed to ensure that the affine independence property required in part 4 of Lemma 2 is maintained along the way, by invoking a version of Caratheodory's Theorem for cones (Theorem 12). In order to effectively execute this strategy, the nested domains of the pseudo-equilibrium equations must also be adjusted, alternating between relinquishing a properness property of the map to permit an application of Sard's Theorem, and subsequently restoring the property so that the set of singular parameters eliminated at each step is closed (and measure zero). A standard application of the Transversality Theorem at the end of this argument is all that is needed to prove Lemma 3.

Lemma 3 may be vacuous as the question of existence of equilibrium is lurking in the background. I will digress to briefly address this question now, as an alternative to the existence arguments in Duggan (2011). Observe that equilibria trivially exist by **(A1)** and **(A2)** if discount factors $\delta_j^z = 0$ for all j, z, as in that case the game is reduced to a sequence of S isolated games of agenda-setting over a compact (joint) set of feasible agreements. Furthermore, regularity in these myopic games amounts to uniqueness of equilibrium as is shown in Lemma 8 in the Appendix. This suggests the construction of a homotopy that affinely traverses the parameter space between a regular game with zero discounting and parameter $\theta_0 \in \Theta$ and a game with the desired level of discounting and some parameter $\theta \in \Theta$. Standard homotopy invariance arguments then ensure that:

Theorem 3. *An equilibrium exists for all $\theta \in \Theta$.*

Note that Theorem 3 delivers existence of equilibria even for non-regular games. One further application of the same homotopy argument this time affinely connecting the regular myopic game to generic parameter θ of the discounted game yields the main result of the analysis:

Theorem 4. *There exists an open, full measure subset of Θ, $\Theta^* \subset \Theta$, such that for all $\theta \in \Theta^*$ the number of equilibria is odd and every equilibrium is regular.*

It immediately follows from an application of the Implicit Function Theorem that:

Corollary 1. *Consider parameters ζ that enter smoothly in equilibrium equations Φ. For all $\theta \in \Theta^*$ every equilibrium outcome distribution is locally expressible as a C^1 function of ζ.*

A frequent criticism of sequential bargaining games is the sensitivity of equilibrium predictions to the bargaining protocol. Theorem 4 ensures that, generically, small changes in the protocol (expressed as changes in state transition probabilities $q(\cdot|z)$) result in small changes on equilibrium outcomes, even allowing smooth comparative statics using standard calculus tools.

8 Variants and special cases

In this section I consider alternative parameterizations that yield variants of the main Theorem 4. In Kalandrakis (2006) I considered regularity of pure strategy equilibria in the space of discount factors of a bargaining model that terminates with agreement. Following the arguments in the proof of Lemma 6 of the appendix,[15] it becomes clear that discount factors δ_j^z can assume the role that preference parameters θ_j^z play in this proof. We can thus consider an alternative parameter space $\widehat{\Theta}$ with elements $\hat{\theta} = (\theta^0, (\delta^z)_{z=1}^Z, (\theta_c)_{c=1}^C) \in \widehat{\Theta} \subset \mathbb{R}^K \times (0,1)^{Z\cdot J} \times \mathbb{R}^{C-S}$, and replace assumption **(A3)** with:

($\mathbf{A3_1}$) Parameters: *For all c, all $x \in Y_c$, all $\hat{\theta}$, and all j,k, $g_k(x,\theta) = g_k(x) + \theta_k^0$ and*

$$f_j(x,c,\hat{\theta}) = \begin{cases} f_j(x,c,\theta_c) & \text{if } c > S, \\ f_j(x,c) & \text{if } c \leq S, \end{cases} \quad \text{and} \quad \frac{\partial f_j(x,c,\theta_c)}{\partial \theta_c} > 0, \text{ if } c > S, j \in J_c^+.$$

In that case:

Theorem 5. *Assume parameter space $\widehat{\Theta}$ instead of Θ and **($\mathbf{A3_1}$)** instead of **(A3)**. There exists an open, full measure subset of $\widehat{\Theta}$, $\widehat{\Theta}^*$, such that for all $\hat{\theta} \in \widehat{\Theta}^*$ the number of equilibria is odd and every equilibrium is regular.*

The conclusion of Theorem 5 is much stronger than Theorem 5 in Kalandrakis (2006) (page 325), even when limited to the model assumed therein. The extra strength of the present results stems from the introduction of coalition/state-specific payoff parameters θ_c.

Turning to these parameters θ_c, note that in both **(A3)** and **($\mathbf{A3_1}$)**, parameter θ_c enters the payoffs of all coalition partners. Tracing the use of this assumption in the proof of the main Theorem, the dependence of coalition partners' payoff on this parameter is necessary only when non-degenerate voter mixing takes place, that is, the set of strictly acceptable proposals is empty. What if we limit attention to *deferrential voting*, such that voters approve proposals whenever they weakly prefer them over the status quo? Such equilibria in pure voting strategies can be defined as follows:

Definition 3. *An* equilibrium with deferrential voting *is an equilibrium σ that satisfies*

$$(DV) \qquad \alpha_j(x,c) = \begin{cases} 1 & \text{if} \quad U_j(x,c;\sigma) \geq U_j(x_s,s;\sigma) \\ 0 & \text{if} \quad U_j(x,c;\sigma) < U_j(x_s,s;\sigma), \end{cases} \qquad \textit{for all } c, \textit{ all } j, \textit{ and all } x.$$

[15] In particular, following *Step 5* of that proof.

Focusing on such equilibria makes it possible to assume that parameters $(\theta_c)_{c>S}$ enter only the corresponding proposers' payoff:

($\mathbf{A3_2}$) **Parameters:** *For all c, all $x \in Y_c$, all θ, and all j,k, $g_k(x,\theta) = g_k(x) + \theta_k^0$ and*

$$f_j(x,c,\theta) = \begin{cases} f_j(x,c,\theta_c) + \theta_j^{z_c}, & j = i_{s_c}, c > S, \\ f_j(x,c) + \theta_j^{z_c}, & \text{otherwise}, \end{cases} \quad \text{and} \quad \frac{\partial f_{i_{s_c}}(x,c,\theta_c)}{\partial \theta_c} \neq 0, \text{ if } c > S.$$

We can now state the following Theorem regarding equilibria with deferrential voting.

Theorem 6. *Assume ($\mathbf{A3_2}$) instead of ($\mathbf{A3}$). There exists an open, full measure subset of Θ, Θ^*, such that for all $\theta \in \Theta^*$ there exists a finite number (possibly zero) of equilibria with deferrential voting and each equilibrium is regular.*

Of course, Theorem 6 has no content when deferrential voting equilibria do not exist. It appears hard to provide a characterization of the environments that admit such equilibria in the general model studied in this paper, but it is possible to derive sufficient conditions. A general, hence not as interesting sufficient condition for existence of equilibria with deferential voting is:

(D0) **Assured Deferrence:** *For all s, all $c \in C_s$, all θ, and all profiles σ, there exists $x \in X_c(\theta)$ such that $U_j(x,c;\sigma) > U_j(x_s,s;\sigma)$ for all $j \in J_c$.*

This condition is likely possible to verify when payoff from status quo is significantly worse compared to available agreements. It is possible to provide readily verifiable conditions in the case of bargaining with termination at agreement. [TO BE ADDED]

9 Legislative Bargaining

So far I have restricted attention to bargaining with *coalitional voting*, that is, in each period the proposal is put to a vote by members of the proposed coalition and behavior of non-coalition members does not affect the outcome in that period. This is appropriate in committee settings where no formal vote is taken, or coalition formation environments where the proposer extracts the consent of coalition partners first before announcing an agreement, as in the case of government formation. In legislative settings though, a proposal takes the form of a policy or bill and the entire legislature may cast a vote, not just a winning coalition put together by the proposer. Importantly, in those settings the behavior of extra-coalition members may affect the outcome, inducing approval by a larger or smaller coalition than the one the proposer intended to build. In such legislative environments, it is more appropriate to assume that following a proposal all players vote and the proposal passes if it is approved by some coalition that is winning among a collection of such decisive coalitions. Fortunately, such legislative environments can be subsumed in the setting of the previous model.[16] To that end, for all s assume there exists an integer $W_s \geq 1$ such that $\{C_s^w\}_{w=1}^{W_s}$ is a partition of C_s.

[16] The key issue here is to ensure that the solution concept used so far amounts, through appropriate transformation, to an equilibrium in this legislative setting.

I now introduce three assumptions that transform the model with coalitional voting into a bargaining model with legislative voting. First, for each s partition the agreement space into W_s subsets each with its own, possibly different across w, voting rule defined as a collection of winning coalitions $\{J_c\}_{c\in C_s^w}, w = 1, \ldots, W_s$.

(L1) Legislative Policies: *For all s, all θ, all distinct w, w', and all distinct c, c', c'', if $c, c' \in C_s^w$, $c'' \in C_s^{w'}$, then $X_c(\theta) = X_{c'}(\theta)$, $z_c = z_{c'}$, $J_c \neq J_{c'}$, and $X_c(\theta) \cap X_{c''}(\theta) = \emptyset$.*

The next assumption requires the introduction of the following restriction on the parameter space:

$$\Theta_L = \{\theta \in \Theta \mid \theta_c = \theta_{c'} \text{ for all } s, w \text{ and all } c, c' \in C_s^w\}.$$

This restricted set of parameters allows us to define preferences over outcomes, this time construed as policies or legislative bills, independent of the voting coalition that approves them.

(L2) Legislative Preferences: *For all s, w, all $c, c' \in C_s^w$, all $\theta \in \Theta_L$, all $x \in X_c(\theta)$, and all j, $f_j(x, c, \theta) = f_j(x, c', \theta)$.*

Finally, we require that the voting rule satisfy a standard monotonicity property, that is, every superset of a decisive coalition is also a decisive coalition.

(L3) Monotonic Voting Rules: *For all s, w and all $c \in C_s^w$, if there exists $j \notin J_c$, then there exists $c' \in C_s^w$ such that $J_{c'} = J_c \cup \{j\}$.*

It is now possible to define a *legislative equilibrium*. In order to minimize new notation, I maintain the same definition of proposal and voting strategies, but alter the interpretation of a proposal (x, c) or a vote on (x, c) by requiring that voter behavior depend only on the proposed policy x for $c, c' \in C_s^w$, and not on the (irrelevant in this context) second argument of the voters' strategies.[17] With that caveat, let the probability that proposal x may pass with the vote of decisive coalition J_c (only) be:

$$\alpha_L(x, c; \sigma) = \prod_{j\in J_c} \alpha_j(x, c) \prod_{j\notin J_c} (1 - \alpha_j(x, c)),$$

and proceed to the definition.

Definition 4. *A* legislative equilibrium *is a profile of strategies σ that satisfies (V) and*

- $\alpha_j(x, c) = \alpha_j(x, c')$, *for all s, w, all $c, c' \in C_s^w$, and all $j \neq i_s$, and*
- $\pi_s(\arg\max_{(x,c)\in\mathbb{R}^D\times\bar{C}_s}\{(\sum_{c'\in C_s^w} \alpha_L(x, c'; \sigma)(U_i(x, c'; \sigma) - U_i(x_s, s; \sigma)) + U_i(x_s, s; \sigma)\}) = 1,$ *for all s.*

[17] Of course, voting can still differ across states s, s' when the same x is proposed.

Say that a legislative equilibrium is equivalent to an equilibrium if it induces the same distribution over policies. We can show the following.

Theorem 7. *Under the additional assumptions (**L1**)-(**L3**), if $\theta \in \Theta_L$, then every equilibrium σ is equivalent to some legislative equilibrium σ^L.*

The main insight behind Theorem 7 is that in the original equilibrium with coalitional voting, if non-degenerate voter mixing prevails for a proposal to some decisive coalition that contains a sub-coalition with members that strictly prefer the proposal and is also decisive, then the proposer must be indifferent between that proposal and the status quo. Thus, we can increase the acceptance probability of that proposed agreement to one while at the same time increasing the probability with which the proposer proposes the status quo, to obtain an equivalent legislative equilibrium. An immediate consequence of Theorems 3 and 7 is that legislative equilibria exist.

Theorem 8. *Under the additional assumptions (**L1**)-(**L3**), a legislative equilibrium exists for all $\theta \in \Theta_L$.*

Unfortunately, given the assumed parameter space we cannot, in general, obtain an analogue of Theorem 4 for legislative equilibria. For general voting rules and continuous policy spaces, there may exist distinct optimal proposals (corresponding to distinct legislative coalitions) drawn from the same feasible set $X_c(\theta) = X_{c'}(\theta), c, c' \in C_s^w$, that leave the proposer indifferent at some state s, and assumption (**L2**) implies that there is not enough richness in the manner parameters enter the model to avoid singularities in the corresponding indifference equations of the proposer. Nonetheless, Theorems 9 and 10 identify two special cases where this problem does not surface. Note that in both of these Theorems, Θ_L is viewed as a $(W - S) + JZ + K$-dimensional set[18] and full measure refers to Lebesgue measure of that dimension. The first case is when each feasible set is a singleton.

Theorem 9. *Under the additional assumptions (**L1**)-(**L3**), if $X_c(\theta)$ is a singleton for all $\theta \in \Theta_L$ then there exists an open, full measure subset of Θ_L, Θ_L^*, such that for all $\theta \in \Theta_L^*$ there exists a finite number (at least one) of legislative equilibrium outcome distributions.*

Theorem 9 provides a gauge of the strength of earlier results when it comes to the richness of the parameter space assumed. In particular, Kalandrakis (2014) shows that in a finite environment as assumed in Theorem 9, if we remove parameters θ_c from the specification of payoffs, then there exists an open set of parameters $(\theta_z)_{z=1}^Z$ that induce a continuum of outcome distributions and payoffs.

The second special case where we can be assured of determinacy of legislative equilibrium outcome distributions is when the voting rule is oligarchic. Voting rule $\{J_c\}_{c \in C_s^w}$ is oligarchic if there exists c such that $J_c \subset J_{c'}$ for all $c' \in C_s^w \setminus \{c\}$.

Theorem 10. *Under the additional assumptions (**L1**)-(**L3**), if for all s, w the voting rule $\{J_c\}_{c \in C_s^w}$ is oligarchic, then there exists an open, full measure subset of Θ_L, Θ_L^*, such that for all $\theta \in \Theta_L^*$ there exists an odd number of legislative equilibrium outcome distributions.*

[18] Due to the restrictions across coordinates of Θ_L. In fact, we can assume $K = 0$ in Theorem 9.

Note that in the above theorem policy sets can be continuous and we recover the full strength of Theorem 4.

To conclude this section, we now return to the special case of deferrential voting. Deferrential voting equilibria make it possible to consider a richer notion of legislative equilibrium such that the proposer's payoff depends not only on the policy that passes but also on the voting coalition that approved it, via payoff parameters θ_c that, when **(A3$_2$)** is assumed instead of **(A3)**, only enter the payoff function of the proposer. We need not provide a new definition for this model, as the definition of legislative equilibrium already anticipates this possibility. But in order to consider this environment, assumption **(L2)** must be revised by imposing a preference for larger legislative voting coalitions on the proposer.

(L2$_1$) **Legislative Preferences:** *For all* s, w, *all* $c, c' \in C_s^w$, *all* $\theta \in \Theta_L$, *all* $x \in X_c(\theta)$, $f_j(x, c, \theta) = f_j(x, c', \theta)$ *for all* $j \neq i_s$ *and if* $J_{c'} \subset J_c$ *then* $f_{i_s}(x, c, \theta) \geq f_{i_s}(x, c', \theta)$.

Without the monotonicity assumption in **(L2$_1$)**, a proposer may find it best to optimize over policies while trying to induce some voters to reject her proposal. The corresponding optimization program, though, would fail the P-Invexity assumption **(A5)**, and may induce multiple solutions or solutions that are not characterized by established equilibrium equations. With **(L2$_1$)** it is possible to state the following result concerning legislative equilibria with deferrential voting.

Theorem 11. *Assume* **(A3$_2$)** *instead of* **(A3)** *and* **(L1)**, **(L2$_1$)**, *and* **(L3)**. *There exists an open, full measure subset of* Θ, Θ^*, *such that for all* $\theta \in \Theta^*$ *there exists a finite number (possibly zero) of legislative equilibria with deferrential voting.*

Once more, Theorem 11 appears vacuous because existence of equilibria with deferrential voting is not guaranteed in general. But there are some important special environments such as generalizations of the legislative bargaining models of Banks and Duggan (2000, 2006) where the extra assumption **(L2$_1$)** along with the parameterization **(A3$_2$)** can be reconciled with existence of equilibria with deferrential voting. In those cases, Theorem 11 provides a sound basis for empirical applications building on such models.

10 Conclusions

I have studied a class of dynamic sequential bargaining models of coalition formation with complete information and established that generically in a relatively austere space of parameters the exists an odd number of equilibria that satisfy a strong form of continuity with respect to model parameters. The result extends to determinacy of outcome distribution in a number of legislative contexts. These findings lay the foundations for empirical applications of these models.

APPENDIX A: Proof of main results

In this appendix, I restate and prove Lemmas 1-3 and prove Theorems 3 and 4. In the process I prove additional Lemmas. The proof of Theorems 1 and 2 follows standard arguments relying on Lemma 1 and is omitted.

Lemma 1 (Restated)**.** *For all $c > S$ and all strategy profiles σ*

1. *x solves $P(c;\sigma)$ if and only if there exist $b = (b_r)_{r\in J_c^+ \cup K_c \cup L_c}$ such that*

$$F_{J_c^+}(x,c,\theta)\cdot[b_{J_c^+}]_+ + G_{K_c}(x,\theta)\cdot[b_{K_c}]_+ + H_{L_c}(x)\cdot b_{L_c} = 0 \tag{2a}$$

$$U_{J_c^-}(x,c;\sigma) - U_{J_c^-}(x_{s_c},s_c;\sigma) - [-b_{J_c^-}]_+ = 0 \tag{2b}$$

$$g_{K_c}(x,\theta) - [-b_{K_c}]_+ = 0 \tag{2c}$$

$$h_{L_c}(x) = 0 \tag{2d}$$

$$\mathbf{1}^T\cdot[b_{J_c^+}]_+ - 1 = 0. \tag{2e}$$

2. *If $\bar{A}(c;\sigma) \neq \emptyset$ then there is a unique x that solves $P(c;\sigma)$.*

3. *There exist x, b, with $[b_i]_+ = 0$, that solve (2) if and only if $\bar{A}(c;\sigma) = \{x\}$ and $A(c;\sigma) = \emptyset$.*

4. *For all $x \in \bar{A}(c;\sigma)$, if $\bar{A}(c;\sigma) \neq \{x\}$ then there exists $\tau > 0$ and a C^1 function $\gamma : [0,\tau] \to \mathbb{R}^D$ such that $\gamma(0) = x$ and $\gamma(t) \in A(c;\sigma)$ for all $t \in (0,\tau]$.*

Proof. First note that (2a)-(2d) and $([b_{J_c^+}]_+, [b_{K_c}]_+, b_{L_c}) \neq 0$ are equivalent to the generalized Fritz-John conditions and are necessary for x to solve $P(c;\sigma)$ (Mangasarian and Fromovitz (1967)). By **(A4)**, if the generalized Fritz-John necessary conditions are met, then $[b_{J_c^+}]_+ \neq 0$ at any solution x, b of (2a)-(2d). It follows that (2) including the normalization (2e) are necessary conditions for x to solve $P(c;\sigma)$. It remains to show that these conditions are sufficient and that any x that satisfies them is the unique solution of $P(c;\sigma)$. In particular, we will show that if there exist x and b that satisfy (2), then there does not exist $x' \in \bar{A}(c;\sigma)$ such that $x \neq x'$ and $f_i(x',c,\theta) \geq f_i(x,c,\theta)$. Assume the existence of such an x' to get a contradiction. Let $\hat{J} = \{j \in J_c^+ \mid [b_j]_+ > 0\}$. From $x' \in \bar{A}(c;\sigma)$ and (2b) we conclude that $\hat{J} \subseteq J_c(x,x',\theta)$. It follows that, by **(A5)**, there exists λ such that $\lambda^T \cdot F_{\hat{J}}(x,c,\theta) > 0, \lambda^T \cdot G_{K_c(x,\theta)}(x,\theta) \geq 0$, and $\lambda^T \cdot H_{L_c}(x) = 0$. But then (2a) and (2e) contradict Theorem 11. Thus, there does not exist $x' \in \bar{A}(c;\sigma)$ such that $x \neq x'$ and $f_i(x',c,\theta) \geq f_i(x,c,\theta)$. This completes the proof of part 1, and establishes the uniqueness of any solution in part 2. The existence of some solution whenever $\bar{A}(c;\sigma) \neq \emptyset$ follows from compactness of $\bar{A}(c;\sigma)$ and continuity of f_i due to **(A1)** and **(A2)**. To show part 3, suppose x, b solve (2) with $[b_i]_+ = 0$ but there exists $x' \neq x$ such that $x' \in \bar{A}(c;\sigma)$. Once more, $x' \in \bar{A}(c;\sigma)$ and (2b) implies that $f_j(x',c,\theta) \geq f_j(x,c,\theta)$ for all $j \in \hat{J} = \{j \in J_c^+ \mid [b_j]_+ > 0\}$ and, by **(A5)**, there exists λ such that $\lambda^T \cdot F_{\hat{J}}(x,c,\theta) > 0, \lambda^T \cdot G_{K_c(x,\theta)}(x,\theta) \geq 0$, and $\lambda^T \cdot H_{L_c}(x) = 0$, again contradicting Theorem 11 since (2a) and (2e) hold. Thus, $\bar{A}(c;\sigma) = \{x\}$. If, in addition, $A(c;\sigma) = \{x\}$, then (2b) imply that $[b_j]_+ = 0$ for all $j \in J_c^-$ which contradicts (2e) since $[b_i]_+ = 0$. Thus, $A(c;\sigma) = \emptyset$ as we wished to show. Lastly, we show part 4. Assume there exist distinct $x, x' \in \bar{A}(c;\sigma)$. By **(A5)** there exists λ such that $\lambda^T \cdot F_{J_c(x,x',\theta)}(x,c,\theta) > 0$, $\lambda^T \cdot G_{K_c(x,\theta)}(x,\theta) \geq 0$, and $\lambda^T \cdot H_{L_c}(x) = 0$.

Let $\hat{J} = \{j \in J_c^- \mid U_j(x, c; \sigma) = U_j(x_{s_c}, s_c; \sigma)\}$ and $\hat{K} = \{k \in K_c(x,\theta) \mid \lambda^T \cdot G_k(x,\theta) = 0\}$. Since both $x, x' \in \bar{A}(c;\sigma)$, then $\hat{J} \subseteq J_c(x, x', \theta)$. Furthermore, $U_j(x, c; \sigma) > U_j(x', c; \sigma) \geq U_j(x_{s_c}, s_c; \sigma)$ for all $j \in J_c^- \setminus J_c(x, x', \theta)$, by the definition of $\hat{J}$ and $J_c(x, x', \theta)$. By **(A4)**, $[G_{\hat{K}}(x,\theta)\ H_{L_c}(x)]$ has full column rank. Then by Lemma 2.1 of Still and Streng (1996) there exists $\tau > 0$ and a C^1 function $\gamma : [0, \tau] \to \mathbb{R}^D$ such that $\gamma(0) = x$, $D_t\gamma(0) = \lambda$, and $\gamma([0, \tau]) \subset \bar{A}(c;\sigma)$ (in fact, with inequality constraints $g_{\hat{K}}$ treated as equality constraints). The fact that $\gamma((0, \tau]) \subset A(c;\sigma)$ follows by choosing small enough τ, since $\lambda^T \cdot F_{J_c(x,x',\theta)}(x, c, \theta) > 0$ and $U_j(x, c; \sigma) > U_j(x_{s_c}, s_c; \sigma)$ for all $j \in J_c^- \setminus J_c(x, x', \theta)$. ☐

Lemma 2 (restated)**.** *If σ is a regular equilibrium then there exists a unique e^* such that $\sigma = \hat{\sigma}(e^*)$ and $\Phi(e^*, \theta) = 0$. Furthermore, e^* is such that:*

1. *$p_c^* \neq 0$ for all c.*
2. *For all $c > S$, $a_c^* \notin \{0, 1\}$ and if $a_c^* \in (0, 1)$ then $p_c^* > 0$.*
3. *$b_{c,r}^* \neq 0$ for all $c > S$ and all $r \in J_c^- \cup K_c$.*
4. *The matrix $\begin{pmatrix} F_{j \in J_c^+ : b_{c,j}^* > 0}(x_c^*, c, \theta) & G_{k \in K_c : b_{c,k}^* > 0}(x_c^*, \theta) & H_{L_c}(x_c^*) \\ \mathbf{1}^T & 0 & 0 \end{pmatrix}$ has full column rank for all $c > S$.*

Proof. By the definition of regularity, if σ is regular then there exists e^* such that $\sigma = \hat{\sigma}(e^*)$, $\Phi(e^*, \theta) = 0$ and $\text{rank}(D_e\Phi(e^*, \theta)) = \dim(E)$. In the next five steps, we obtain implications 1-4 of the Lemma, and then e^* is shown to be unique.

Step 1: *$p_c^* \neq 0$ for all c.* If $p_c^* = 0$ for some c, then $\frac{\partial [p_c^*]_+}{\partial p_c} = \frac{\partial [-p_c^*]_+}{\partial p_c} = 0$ and $D_{p_c}\Phi(e^*, \theta) = 0$, contradicting the full rank of $D_e\Phi(e^*, \theta)$.

Step 2: *$a_c^* \notin \{0, 1\}$ for all $c > S$.* If $a_c^* \in \{0, 1\}$ for some c, then $\frac{\partial [a_c^*]_0^1}{\partial a_c} = \frac{\partial [-a_{c'}]_+}{\partial a_{c'}} = \frac{\partial [a_c^* - 1]_+}{\partial a_c} = 0$ and $D_{a_c}\Phi(e^*, \theta) = 0$, contradicting the full rank of $D_e\Phi(e^*, \theta)$.

Step 3: *If $a_c^* \in (0, 1)$ for some $c > S$, then $p_c^* > 0$.* Fix $c > S$ such that $a_c^* \in (0, 1)$. By virtue of *Step 1*, it suffices to show that it cannot be that $p_c^* < 0$. Assume $p_c^* < 0$ to get a contradiction. Then, since $[p_c^*]_+ = \frac{\partial [p_c^*]_+}{\partial p_c} = \frac{\partial [-a_c^*]_+}{\partial a_c} = \frac{\partial [a_c^* - 1]_+}{\partial a_c} = 0$,

$$D_{(a_c, p_c)}\Phi(e^*, \theta) = \begin{matrix} \text{(1f)} \\ \textit{remaining} \\ \textit{equations} \end{matrix} \overset{\qquad\qquad a_c \qquad\qquad\qquad\qquad p_c}{\begin{pmatrix} \frac{\partial [a_c^*]_0^1}{\partial a_c}(U_i(e^*, c, \theta) - U_i(e^*, s_c, \theta)) & -\frac{\partial [-p_c^*]_+}{\partial p_c} \\ 0 & 0 \end{pmatrix}} \sim \overset{a_c \qquad p_c}{\begin{pmatrix} 0 & -\frac{\partial [-p_c^*]_+}{\partial p_c} \\ 0 & 0 \end{pmatrix}}$$

so that $D_e\Phi(e^*, \theta)$ is rank-deficient, a contradiction.

Step 4: *$b_{c,r}^* \neq 0$ for all $c > S$ and all $r \in J_c^- \cup K_c$.* If $b_{c,r}^* = 0$ for some $c, r \in J_c^- \cup K_c$, then $\frac{\partial [b_{c,r}^*]_+}{\partial b_{c,r}} = \frac{\partial [-b_{c,r}^*]_+}{\partial b_{c,r}} = 0$ and $D_{b_{c,r}}\Phi(e^*, \theta) = 0$, contradicting the full rank of $D_e\Phi(e^*, \theta)$.

Step 5: *The matrix $\begin{pmatrix} F_{j \in J_c^+ : b_{c,j}^* > 0}(x_c^*, c, \theta) & G_{k \in K_c : b_{c,k}^* > 0}(x_c^*, \theta) & H_{L_c}(x_c^*) \\ \mathbf{1}^T & 0 & 0 \end{pmatrix}$ has full column rank for all $c > S$.* Suppose there exists $c' > S$ such that the rank condition fails. Let $\hat{J}_{c'} = \{j \in J_{c'}^+ :$

$b^*_{c',j} > 0\}$, $\hat{K}_{c'} = \{k \in K_{c'} : b^*_{c',k} > 0\}$, and $y = (b_{c',\hat{J}_{c'}}, b_{c',\hat{K}_{c'}}, b_{L_{c'}})$, and distinguish two cases. Case 1, $i \notin \hat{J}_{c'}$: since $[b_{c',r}]'_+ > 0$ for all $r \in \hat{J}_{c'} \cup \hat{K}_{c'}$ the matrix

$$D_y\Phi(e^*,\theta) = \begin{array}{c} \\ (1a)_{c=c'} \\ (1e)_{c=c'} \\ \textit{remaining equations} \end{array} \begin{array}{c} \begin{array}{ccc} b_{c',\hat{J}_{c'}} & b_{c',\hat{K}_{c'}} & b_{L_{c'}} \end{array} \\ \begin{pmatrix} (F_j(x^*,c',\theta)[b_{c',j}]'_+)_{j\in\hat{J}_{c'}} & (G_k(x^*,\theta)[b_{c',k}]'_+)_{k\in\hat{K}_{c'}} & H_{L_{c'}}(x^*) \\ ([b_{c',\hat{J}_{c'}}]'_+)^T & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

is also rank-deficient, implying $D_e\Phi(e^*,\theta)$ is rank deficient, a contradiction. Case 2, $i \in \hat{J}_{c'}$: let $\hat{J}^-_{c'} = \hat{J}_{c'} \setminus \{i\}$. After dividing the column that corresponds to $b_{c',r}$ by $[b_{c',r}]'_+ > 0$ for all $r \in \hat{J}_{c'} \cup \hat{K}_{c'}$, multiplying the column that corresponds to $a_{c'}$ with $\frac{1}{\frac{\partial [a^*_{c'}-1]_+}{\partial a_{c'}}[b_{c',i}]'_+}$ and adding the product to the column that corresponds to $b_{c',i}$ (noting that $\frac{\partial [a^*_{c'}]^1_0}{\partial a_{c'}} = 0$), we conclude that $D_{(a_{c'},y)}\Phi(e^*,\theta)$ has the same rank as:

$$\begin{array}{c} \\ (1a)_{c=c'} \\ (1e)_{c=c'} \\ (1g)_{c=c'} \\ \textit{remaining equations} \end{array} \begin{array}{c} \begin{array}{ccccc} a_{c'} & b_{c',i} & b_{c',\hat{J}^-_c} & b_{c',\hat{K}_c} & b_{L_{c'}} \end{array} \\ \begin{pmatrix} 0 & F_i(x^*,c',\theta) & F_{\hat{J}^-_{c'}}(x^*,c',\theta) & G_{\hat{K}_{c'}}(x^*,\theta) & H_{L_{c'}}(x^*) \\ 0 & 1 & \mathbf{1}^T & 0 & 0 \\ -\frac{\partial [a^*_{c'}-1]_+}{\partial a_{c'}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

once more contradicting the full rank of $D_e\Phi(e^*,\theta)$.

It remains to show that e^* is unique. Suppose not, i.e., suppose there exists $e' \neq e^*$ such that $\hat{\sigma}(e^*) = \hat{\sigma}(e') = \sigma$ and $\Phi(e',\theta) = 0$. Since $\hat{\sigma}(e^*) = \hat{\sigma}(e')$, we have $x^*_c = x'_c$ and $a^*_c = a'_c$, and $p^*_c = p'_c > 0$ for all c proposed with positive probability in σ, and $V^*_{z,j} = V'_{z,j} = V_j(z;\sigma)$ for all z, j. It follows that for the remaining c such that $\bar{A}(c;\sigma) \neq \emptyset$, we must have $x^*_c = x'_c$ (since $V^*_{z,j} = V'_{z,j}$ for all z, j and part 2 of Lemma 1), and $p^*_c < 0$, by *Step 1*, and $p'_c \leq 0$. Furthermore, it cannot be that $a^*_c \leq 1$ by *Steps 2,3*, so $a'_c \geq 1$. But then $p^*_c = p'_c$ by equation (1f). To summarize:

Step 6: *If $e^* \neq e'$ satisfy $\hat{\sigma}(e^*) = \hat{\sigma}(e') = \sigma$ and $\Phi(e^*,\theta) = \Phi(e',\theta) = 0$, then $V^*_{z,j} = V'_{z,j} = V_j(z;\sigma)$ for all z, j, and $x^*_c = x'_c$, and $p^*_c = p'_c$ for all c such that $\bar{A}(c;\sigma) \neq \emptyset$. Furthermore, $a^*_c = a'_c$ if $p^*_c > 0$ and $a^*_c > 1$ and $a'_c \geq 1$ if $p^*_c < 0$ and $\bar{A}(c;\sigma) \neq \emptyset$.*

Next we show:

Step 7: *If $e^* \neq e'$ satisfy $\hat{\sigma}(e^*) = \hat{\sigma}(e') = \sigma$ and $\Phi(e^*,\theta) = \Phi(e',\theta) = 0$, then $x^*_c = x'_c$, $a^*_c = a'_c < 0$, and $p^*_c = p'_c < 0$ for all c such that $\bar{A}(c;\sigma) = \emptyset$.* It cannot be that $a^*_c, a'_c \geq 0$ for in that case *Step 6* and equations (1b)-(1d) imply $x^*_c, x'_c \in \bar{A}(c;\sigma)$. If $a^*_c = a'_c < 0$ and $x^*_c = x'_c$, then indeed $p^*_c = p'_c < 0$ by equation (1f), as we wish to show. So consider $(a^*_c, x^*_c) \neq (a'_c, x'_c)$, and assume, without loss of generality, that $a^*_c \leq a'_c < 0$. By equations (1b) and *Step 6* we conclude that for all

$j \in \hat{J} = \{j \in J_c \mid b^*_{c,j} > 0\}$

$$\begin{aligned} f_j(x^*_c, c, \theta) &= -[-a^*_c]_+ - \delta_j^{z_c} V^*_{z_c,j} + U_j(e^*, s_c, \theta) \\ &\leq -[-a'_c]_+ - \delta_j^{z_c} V'_{z_c,j} + U_j(e', s_c, \theta) + [-b'_{c,j}]_+ = f_j(x'_c, c, \theta), \end{aligned}$$

with strict inequality if $a^*_c \neq a'_c$. Thus, since $(a^*_c, x^*_c) \neq (a'_c, x'_c)$, it follows that $x'_c \neq x^*_c$ and $\hat{J} \subseteq J_c(x^*, x', \theta)$. So by **(A5)** we conclude that there exists λ such that $\lambda^T \cdot F_{\hat{J}}(x^*_c, c, \theta) > 0$, $\lambda^T \cdot G_{K_c(x^*_c,\theta)}(x^*_c, \theta) \geq 0$, and $\lambda^T \cdot H_{L_c}(x^*_c) = 0$. But by Theorem 11, b^*_c does not solve equation (1a), a contradiction.

Step 8: *If $e^* \neq e'$ satisfy $\hat{\sigma}(e^*) = \hat{\sigma}(e') = \sigma$ and $\Phi(e^*, \theta) = \Phi(e', \theta) = 0$, then $b^*_c = b'_c$ for all c, and $a^*_c = a'_c > 1$ for c such that $p^*_c < 0$ and $\bar{A}(c; \sigma) \neq \emptyset$.* First, $b^*_{c,i} = b'_{c,i}$ for all c, except (possibly) c such that $a^*_c > 0$, $p^*_c < 0$ and $\bar{A}(c; \sigma) \neq \emptyset$, by equation (1g) and *Step 6,7*. Furthermore, $b^*_{c,r} = b'_{c,r}$ for all $c, r \in J^-_c \cup K_c$ such that $b^*_{c,r} < 0$, by equations (1b) and *Step 6,7*. Since $b^*_{c,r} \neq 0$ for all $r \in J^-_c \cup K_c$, by *Step 4*, it remains to show $[b^*_{c,\hat{J}_c}]_+ = [b'_{c,\hat{J}_c}]_+$, $[b^*_{c,\hat{K}_c}]_+ = [b'_{c,\hat{K}_c}]_+$, and $b^*_{L_c} = b'_{L_c}$, where $\hat{J}_c = \{j \in J_c \mid b^*_{c,j} > 0\}$ and $\hat{K}_c = \{j \in J_c \mid b^*_{c,j} > 0\}$. Suppose not. Then combining equations (1a) and (1e) we conclude that $\lambda^T = (([b^*_{c,\hat{J}_c}]_+ - [b'_{c,\hat{J}_c}]_+)^T, ([b^*_{c,\hat{K}_c}]_+ - [b'_{c,\hat{K}_c}]_+)^T, (b^*_{L_c} - b'_{L_c})^T) \neq 0$ satisfies

$$\begin{pmatrix} F_{\hat{J}_c}(x^*, c, \theta) & G_{\hat{K}_c}(x^*, \theta) & H_{L_c}(x^*) \\ \mathbf{1}^T & 0 & 0 \end{pmatrix} \cdot \lambda^T = 0,$$

contradicting *Step 5*. This also proves $a^*_c = a'_c > 1$ if $p^*_c < 0$ and $\bar{A}(c; \sigma) \neq \emptyset$ by *Step 6* and equation (1g).

Jointly, *Steps 6-8* establish the uniqueness of e^*. □

In what follows I construct a sequence of pseudo-equilibrium maps that isolate singularities of equilibrium equations $\Phi(e, \theta) = 0$ (i.e., equations (1)) and the subset of parameters for which such singularities are possible. The upshot of this construction is the conclusion of Lemma 7, that is, that such singularities cannot occur in an open full measure subset of parameters. To start, define

$$\mathcal{I} = \left\{ \left((\mathring{C}^p_s)_{s=1}^S, (\mathring{C}^a_s)_{s=1}^S, (\mathring{J}_c)_{c>S}, (\mathring{K}_c)_{c>S} \right) \middle| \begin{array}{l} \mathring{C}^p_s \subseteq C_s \text{ and } \mathring{C}^a_s \subseteq C_s \text{ for all } s, \\ \mathring{J}_c \subseteq J^-_c \text{ and } \mathring{K}_c \subseteq K_c \text{ for all } c \end{array} \right\},$$

and partition $\mathcal{I}$ into $M+1$ susbsets $\mathcal{I}_0, \ldots, \mathcal{I}_m, \ldots, \mathcal{I}_M$ where $M = 2\sum_{s=1}^S \#C_s + \sum_{c=S+1}^C (\#J^-_c + \#K_c)$ and

$$\mathcal{I}_m = \left\{ \mu \in \mathcal{I} \middle| \sum_{s=1}^S (\#\mathring{C}^p_s + \#\mathring{C}^a_s) + \sum_{c>S} (\#\mathring{J}_c + \#\mathring{K}_c) = m \right\}.$$

I use the notation $\mathring{C}_s = \mathring{C}^p_s \cup \mathring{C}^a_s$ and $\dot{C}_s = \mathring{C}^p_s \cap \mathring{C}^a_s$. Further for all $\mu \in \mathcal{I}$ define the index set

$$N(\mu) = \{(s, c) \mid c \in \mathring{C}_s, s = 1, \ldots, S\} \cup \{(c, r) \mid r \in \mathring{J}_c \cup \mathring{K}_c, c = S+1, \ldots, C\} \cup \{0\},$$

and for all $\mu \in \mathcal{I}$ and all $\nu \in N(\mu)$ define the system of equations:

$$\begin{aligned}
&(3a) && F_{J_c^+}(x_c, c, \theta) \cdot [b_{c,J_c^+}]_+ + G_{K_c}(x_c, \theta) \cdot [b_{c,K_c}]_+ + H_{L_c}(x_c) \cdot b_{c,L_c} = 0 \\
&(3b) && U_{c,J_c^- \setminus \mathring{J}_c}(e, \theta) - U_{s_c, J_c^- \setminus \mathring{J}_c}(e, \theta) - [-b_{c,J_c^- \setminus \mathring{J}_c}]_+ + \mathbf{1} \cdot [-a_c]_+ = 0 \\
&(3\mathring{b}) && b_{c,\mathring{J}_c} = 0 \\
&(3c) && g_{K_c \setminus \mathring{K}_c}(x_c, \theta) - [-b_{c,K_c \setminus \mathring{K}_c}]_+ = 0 \\
&(3\mathring{c}) && b_{c,\mathring{K}_c} = 0 \\
&(3d) && h_{L_c}(x_c) = 0 \\
&(3e) && \mathbf{1}^T \cdot [b_{c,J_c^+}]_+ - 1 = 0 \\
&(3\mathring{f}_p) && p_c = 0, \text{ if } c \in \mathring{C}_s^p \\
&(3\mathring{f}_a) && a_c - 1 = 0, \text{ if } c \in \mathring{C}_s^a \\
&(3f) && [-p_s]_+ - [a_c]_0^1 (U_i(e, c, \theta) - U_i(e, s_c, \theta)) - [-p_c]_+ + [-a_c]_+ = 0, \text{ if } c \notin \mathring{C}_s \\
&(3g) && b_{c,i} - [a_c - 1]_+ = 0,
\end{aligned}$$

for all s and all $c \in C_s$,

$$(3h) \qquad \sum_{c \in \bar{C}_s} [p_c]_+ - 1 = 0,$$

for all s,

$$(3i) \qquad V_j^z - \sum_{s=1}^{S} q(s|z) \sum_{c \in \bar{C}_s} [p_c]_+ ([a_c]_0^1 U_j(e, c, \theta) + (1 - [a_c]_0^1) U_j(e, s, \theta)) = 0,$$

for all z, j, and

$$(3\nu) \qquad \left.\begin{aligned} [-p_s]_+ - [a_c]_0^1 (U_i(e, c, \theta) - U_i(e, s, \theta)) - [-p_c]_+ + [-a_c]_+ \\ U_j(e, c, \theta) - U_j(e, s_c, \theta) - [-b_{c,j}]_+ + [-a_c]_+ \\ g_k(x_c, \theta) - [-b_{c,k}]_+ \end{aligned}\right\} = 0 \begin{cases} \nu = (s, c), c \in \mathring{C}_s, \\ \nu = (c, j), j \in \mathring{J}_c, \\ \nu = (c, k), k \in \mathring{K}_c, \end{cases}$$

if $\nu \neq 0$.

Equations (3) corresponding to $\mu = \left((\mathring{C}_s^p)_{s=1}^S, (\mathring{C}_s^a)_{s=1}^S, (\mathring{J}_c)_{c>S}, (\mathring{K}_c)_{c>S}\right)$ replace acceptance and inequality constraints (1b) and (1c) with $b_{c,r} = 0$ for all c and $r \in \mathring{J}_c \cup \mathring{K}_c$ and substitute $p_c = 0$ for all s and $c \in \mathring{C}_s^p$ and $a_c = 1$ for all s and $c \in \mathring{C}_s^a$ instead of the corresponding equation (1f). Equation (3ν) restores one of the replaced equations corresponding to $\nu \in N(\mu)$, if $\nu \neq 0$. Note that the total number of equations in system (3) corresponding to μ, ν is given by

$$\#(\mu, \nu) = \dim(E) + \sum_{s=1}^{S} \#\dot{C}_s + \mathbb{I}(\nu \neq 0).$$

Now for all $m = 0, 1, \ldots, M$, all $\mu \in \mathcal{I}_m$, and all $\nu \in N(\mu)$ define a function $\Phi^{\nu}_{m,\mu} : \Xi_m \to \mathbb{R}^{\#(\mu,\nu)}$ as the left-hand-side of equations (3). To define the domains $\Xi_m, m = 0, 1, \ldots, M$, of these functions, for any $\varepsilon > 0$ let $X^{\varepsilon}_c = \bigcup_{x \in X_c} \{y \in \mathbb{R}^D \mid ||x - y|| < \varepsilon\}$ and let $\bar{X}^{\varepsilon}_c$ denote its closure. Recall that $Y_c = X^{\bar{\varepsilon}}_c$ for $\bar{\varepsilon} > 0$. Fix a sequence $\{\varepsilon_m\}^{M+1}_{m=0}$ satisfying $\bar{\varepsilon} = \varepsilon_{M+1} > \varepsilon_M > \ldots > \varepsilon_m > \ldots > \varepsilon_1 > \varepsilon_0 = 0$. For all $\theta \in \Theta$ define

$$\bar{u}(\theta) = \frac{1}{1 - \max_{z,j} \delta^z_j} \max\{f_j(x, c, \theta) \mid c = 1, \ldots, C, x \in X_c, j = 1 \ldots, J\},$$

which is a continuous function of θ by the Theorem of the Maximum.[19] Ξ_m and corresponding restrictions $\bar{\Xi}_m, m = 0, 1, \ldots, M$, are inductively defined as

$$\Xi_m = \left\{ (e, \theta) \in E \times \Theta_m \;\middle|\; [-p_c]_+ < \begin{cases} \bar{u}(\theta) + \varepsilon_{m+1} \text{ for all } c \le S \\ 2(\bar{u}(\theta) + \varepsilon_{m+1}) \text{ for all } c > S \end{cases}, \text{ and } x_c \in X^{\varepsilon_{m+1}}_c \text{ for all } c > S \right\},$$

$$\bar{\Xi}_m = \left\{ (e, \theta) \in E \times \Theta_m \;\middle|\; [-p_c]_+ \le \begin{cases} \bar{u}(\theta) + \varepsilon_m \text{ for all } c \le S \\ 2(\bar{u}(\theta) + \varepsilon_m) \text{ for all } c > S \end{cases}, \text{ and } x_c \in \bar{X}^{\varepsilon_m}_c \text{ for all } c > S \right\}.$$

The sets Θ_m (hence $\Xi_m, \bar{\Xi}_m$) are obtained as

$$\Theta_m = \Theta \setminus \left(\bigcup_{m' > m} \bigcup_{\mu \in \mathcal{I}_{m'}} \bigcup_{\substack{\nu \in N(\mu): \\ \#(\mu,\nu) > \dim(E)}} \Theta^{\nu}_{m',\mu} \right), m = 0, 1, \ldots, M,$$

where for all $m = 0, 1, \ldots, M$, all $\mu \in \mathcal{I}_m$, and all $\nu \in N(\mu)$

$$\Theta^{\nu}_{m,\mu} = \left\{ \theta \in \Theta_m \mid \exists (e, \theta) \in \bar{\Xi}_m \text{ such that } \Phi^{\nu}_{m,\mu}(e, \theta) = 0 \text{ and } \operatorname{rank}(D_e \Phi^{\nu}_{m,\mu}(e, \theta)) < \#(\mu, \nu) \right\}.$$

Note that the condition $\operatorname{rank}(D_e \Phi^{\nu}_{m,\mu}(e, \theta)) < \#(\mu, \nu)$ is trivially met if $\#(\mu, \nu) > \dim(E)$, that is, for those sets $\Theta^{\nu}_{m,\mu}$ that appear in the definition of the sequence $\Theta_m, m = 0, \ldots, M$.

With the goal to show that Θ_m is an open set of full measure in Θ, and ultimately prove Lemma 3, start by establishing that the domain restrictions imposed on the auxiliary functions $\Phi^{\nu}_{m,\mu}$ do not exclude any relevant solutions of the equilibrium mapping Φ.

Lemma 4. $\{e \in E \mid \exists \theta \in \Theta_0$ *such that* $\Phi(e, \theta) = 0\} \subseteq \bar{\Xi}_0 \subset \Xi_0 \subset \bar{\Xi}_1 \subset \Xi_1 \subset \ldots \subset \bar{\Xi}_m \subset \Xi_m$.

Proof. Only the first inclusion needs proof, as the remaining are obvious from the definition of Θ_m and $\bar{\Xi}_m, \Xi_m$. In particular, we need to show that for all $\theta \in \Theta_0$, and any solution e of $\Phi(e, \theta) = 0$, $(e, \theta) \in \bar{\Xi}_0$. Recall that $\epsilon_0 = 0$ so that

$$\bar{\Xi}_0 = \left\{ (e, \theta) \in E \times \Theta_0 \mid [-p_c]_+ \le \begin{cases} \bar{u}(\theta) & \text{for all } c \le S, \\ 2\bar{u}(\theta) & \text{for all } c > S, \end{cases} \text{ and } x_c \in X_c \text{ for all } c > S \right\}.$$

[19] In particular, the compact constraint set does not depend on θ.

Indeed, $x_c \in X_c$ by **(A1)**. Furthermore, $[-p_s]_+ \leq \bar{u}(\theta)$ for all s, by equation (1f). In particular, it must be that $[p_c]_+ > 0$ for some $c \in \bar{C}_s$ by equation (1h), so $[-p_s]_+ = 0$, if $c = s$, or $[-p_s]_+ \leq [a_c]_0^1(U_i(e, c, \theta) - U_i(e, s_c, \theta)) \leq \bar{u}(\theta)$, if $c \neq s$. It remains to show that $[-p_c]_+ \leq 2\bar{u}(\theta)$ for all $c > S$. Equation (1f) now implies $[-p_c]_+ \leq [-p_s]_+ + [-a_c]_+$, and it suffices to show that $[-a_c]_+ \leq \bar{u}(\theta)$. To that end, note that $[b_{c,j}]_+ > 0 = [-b_{c,j}]_+$ for some $j \in J_c^+$ by (1e), so that either $a_c > 1$ if $j = i_s$ (by (1g)) or $[-a_c]_+ = U_j(e, s, \theta) - U_j(e, c, \theta) \leq \bar{u}(\theta)$ by equation (1b) and $j \in J_c^-$, which establishes the bound $[-p_c]_+ \leq 2\bar{u}(\theta)$ for all $c > S$. □

Set $\Theta^\nu_{m,\mu}$ is defined as a subset of parameters that appear in the intersection of a subset of the pre-image of $\Phi^\nu_{m,\mu}$ and a restricted subset, $\bar{\Xi}_m$, of the domain of $\Phi^\nu_{m,\mu}$. This second restriction ensures that these sets are (relatively) closed:

Lemma 5. *For all $m = 0, 1, \ldots, M$, all $\mu \in \mathcal{I}_m$, and all $\nu \in N(\mu)$, $\Theta^\nu_{m,\mu}$ is closed in Θ_m.*

Proof. Fix m, $\mu \in \mathcal{I}_m$, and $\nu \in N(\mu)$ and consider a sequence θ^n in $\Theta^\nu_{m,\mu} \subset \Theta_m$ that converges to $\theta^* \in \Theta_m$. We wish to show that $\theta^* \in \Theta^\nu_{m,\mu}$. Since $\theta^n \in \Theta^\nu_{m,\mu} \subset \Theta_m$, for all n there exists e^n such that $(e^n, \theta^n) \in \bar{\Xi}_m$, $\Phi^\nu_{m,\mu}(e^n, \theta^n) = 0$, and $\operatorname{rank}(D_e\Phi^\nu_{m,\mu}(e^n, \theta^n)) < \#(\mu, \nu)$. Since $\Phi^\nu_{m,\mu}$ is C^1, if there exists a subsequence such that $e^n \to e^*$ and $(e^*, \theta^*) \in \bar{\Xi}_m$, then $\Phi^\nu_{m,\mu}(e^*, \theta^*) = 0$, and $\operatorname{rank}(D_e\Phi^\nu_{m,\mu}(e^*, \theta^*)) < \#(\mu, \nu)$ (by continuity of the determinant), that is, $\theta^* \in \Theta^\nu_{m,\mu}$. Then it suffices to show that there exists a subsequence such that $e^n \to e^*$ and $(e^*, \theta^*) \in \bar{\Xi}_m$. We proceed in a series of steps, each step identifying a finer subsequence compared to the previous step. We index all successive subsequences by n to avoid notational clutter.

Step 1: *There exists a subsequence such that $x^n_c \to x^*_c$ for all c.* Since $x^n_c \in \bar{X}_c^{\varepsilon_m}$ for all n, a compact set.

Step 2: *There exists a subsequence such that $[p^n_c]_+ \to \overset{+}{p}_c \in [0, 1]$ for all c, and $\sum_{c \in \bar{C}_s} \overset{+}{p}_c = 1$ for all s.* By equation (3*h*) and compactness of the $(\#C_s - 1)$-dimensional unit simplex in $\mathbb{R}^{\#C_s}$.

Step 3: *There exists a subsequence such that $[a^n_c]_0^1 \to \overset{+}{a}_c$ for all c.* Since $[a^n_c]_0^1 \in [0, 1]$ for all n.

The next two steps do not require further refinement of the sequence. Let $V^n = (V_j^{z,n})_{z,j}$.

Step 4: $V^n \to V^*$. Since $\delta^z_j < 1$ for all z, j, for all n, V^n *uniquely* solves the linear system of equations (3*i*). By *Steps 1-3* V^n converges to the unique solution of the limit system: for all z, j

$$V^z_j - \sum_{s=1}^{S} q(s|z) \sum_{c \in \bar{C}_s} \overset{+}{p}_c(\overset{+}{a}_c(f_j(x^*_c, c, \theta^*) + \delta_j^{z_c} V_{z_c,j}) + (1 - \overset{+}{a}_c)(f_j(x_s, s, \theta^*) + \delta_j^{z_s} V_{z_s,j})) = 0.$$

Step 5: *There exists a subsequence such that $[b^n_{c,j}]_+ \to \overset{+}{b}_{c,j} \in [0, 1]$ for all c and all $j \in J_c^+$, and $\sum_{j \in J_c^+} \overset{+}{b}_{c,j} = 1$ for all c.* By equation (3*e*), and the same argument as in *Step 2*.

Step 6: *$b^n_{c,r} \to b^*_{c,r} = 0$ for all c and all $r \in \mathring{J}_c \cup \mathring{K}_c$, $p^n_c \to p^*_c = 0$ for all $c \in \mathring{C}^p_s$ and all s, and $a^n_c \to a^*_c = 1$ for all $c \in \mathring{C}^a_s$ and all s.* By equations (3$\mathring{b}$), (3$\mathring{c}$), (3$\mathring{f}_a$), and (3$\mathring{f}_p$).

Step 7: *There exists a subsequence such that $[-a^n_c]_+ \to \bar{a}_c$ for all s and all $c \in C_s \setminus \mathring{C}^a_s$.* Distinguish two cases. If $\overset{+}{a}_c > 0$, then by *Step 3* we conclude that $[-a^n_c]_+ \to \bar{a}_c = 0$. If $\overset{+}{a}_c = 0$ instead, then by equation ($3g$) we conclude that $[a_c^n - 1]_+ \to 0$, hence certainly $\overset{+}{b}_{c,i}$ of *Step 5* satisfies $\overset{+}{b}_{c,i} = 0$. Then by *Steps 5-6* there exists $j \in J_c^- \setminus \mathring{J}_c$ such that $\overset{+}{b}_{c,j} > 0$ hence $[-b_{c,j}^n]_+ \to 0$, and by equation ($3b$) and *Steps 1* and *4* it follows that

$$[-a_c^n]_+ \to \bar{a}_c = (f_{c,j}(x_c^*, \theta^*) + \delta_j^{z_c} V_{z_c,j}^*) - (f_{s,j}(x_s, \theta^*) + \delta_j(z_s) V_{z_s,j}^*).$$

Step 8: *There exists a subsequence such that* $[-p_c^n]_+ \to \bar{p}_c \in [0, \bar{u}(\theta^*) + \varepsilon_m]$ *for all* $c \leq S$ *and* $[-p_c^n]_+ \to \bar{p}_c \in [0, 2(\bar{u}(\theta^*) + \varepsilon_m)]$ *for all* $c > S$. Recall that $[-p_c^n]_+ \leq \bar{u}(\theta^n) + \varepsilon_m$ for all $c \leq S$. By the continuity of $\bar{u}(\theta)$, there is a subsequence of $[-p_c^n]_+$ that converges to some $\bar{p}_c \in [0, \bar{u}(\theta^*) + \varepsilon_m]$. We similarly conclude that $[-p_c^n]_+ \to \bar{p}_c \in [0, 2(\bar{u}(\theta^*) + \varepsilon_m)]$ for all $c > S$.

Step 9: $[-b_{c,K_c \setminus \mathring{K}_c}^n]_+ \to \bar{b}_{c,K_c \setminus \mathring{K}_c}$. By equation ($3c$) and Step 1,

$$[-b_{c,K_c \setminus \mathring{K}_c}^n]_+ = g_{K_c \setminus \mathring{K}_c}(x_c^n, \theta^n) \to g_{K_c \setminus \mathring{K}_c}(x_c^*, \theta^*) = \bar{b}_{c,K_c \setminus \mathring{K}_c}.$$

Step 10: *There exists a subsequence such that* $[b_{c,K_c \setminus \mathring{K}_c}^n]_+ \to \overset{+}{b}_{c,K_c \setminus \mathring{K}_c}$ *and* $b_{c,L_c}^n \to b_{c,L_c}^*$ *for all* c. For each n, c, let $\hat{K}_c^n = \{k \in K_c \mid b_{c,k}^n > 0\}$, that is, the subset of inequality constraints with strictly positive multipliers (note that $\hat{K}_c^n \subseteq K_c \setminus \mathring{K}_c$). Going to a subsequence if necessary we conclude that $\hat{K}_c^n \to \hat{K}_c^*$ for all c, since there exists only a finite number of such subsets. So we can assume that $\hat{K}_c^n = \hat{K}_c^*$ for all n, c and it follows that $\hat{K}_c^* \subseteq K_c(x_c^n, \theta^n)$ for all n, c, the set of binding inequality constraints at x_c^n, and that $[b_{c,k}^n]_+ = 0$ for all $k \notin \hat{K}_c^*$ and all n, c. We now recall that equation ($3a$) holds for all n, which we rewrite as

$$F_{c,J_c^+}(x_c^n, \theta^n) \cdot [b_{c,J_c^+}^n]_+ = -[G_{\hat{K}_c^*}(x_c^n, \theta^n) \ H_{L_c}(x_c^n) \ G_{K_c \setminus \hat{K}_c^*}(x_c^n, \theta^n)] \cdot ([b_{c,\hat{K}_c^*}^n]_+^T \ b_{c,L_c}^{n,T} \ [b_{c,K_c \setminus \hat{K}_c^*}^n]_+^T)^T.$$

By continuity of g, constraints $\hat{K}_c^*$ are binding at x_c^*, i.e., $\hat{K}_c^* \subseteq K_c(x_c^*, \theta^*)$, and

$$[G_{\hat{K}_c^*}(x_c^n, \theta^n) \ H_{L_c}(x_c^n) \ G_{K_c \setminus \hat{K}_c^*}(x_c^n, \theta^n)] \to [G_{\hat{K}_c^*}(x_c^*, \theta^*) \ H_{L_c}(x_c^*) \ G_{K_c \setminus \hat{K}_c^*}(x_c^*, \theta^*)],$$

by *Step 1* and **(A2)**. By **(A4)**, $[G_{\hat{K}_c^*}(x_c^*, \theta^*) \ H_{L_c}(x_c^*)]$ has at least $\#\hat{K}_c^* + \#L_c \leq D$ linear independent rows. We write the equations of ($3a$) that correspond to these rows as:

$$F_{c,J_c^+}^{ind}(x_c^n, \theta^n) \cdot [b_{c,J_c^+}^n]_+ = -[G_{\hat{K}_c^*}^{ind}(x_c^n, \theta^n) \ H_{L_c}^{ind}(x_c^n) \ G_{K_c \setminus \hat{K}_c^*}^{ind}(x_c^n, \theta^n)] \cdot ([b_{c,\hat{K}_c^*}^n]_+^T \ b_{c,L_c}^{n,T} \ [b_{c,K_c \setminus \hat{K}_c^*}^n]_+^T)^T,$$

where the superscript *ind* in F, G, H indicates that we have selected the identified $\#\hat{K}_c^* + \#L_c \leq D$ linear independent rows of $[G_{\hat{K}_c^*}(x_c^*, \theta^*) \ H_{L_c}(x_c^*)]$. Notice that $[G_{\hat{K}_c^*}^{ind}(x_c^n, \theta^n) \ H_{L_c}^{ind}(x_c^n)]$ is a square matrix and by continuity of the determinant, there exists some n' such that this matrix has full rank for all $n > n'$. Recalling that $[b_{c,k}^n]_+ = 0$ for all $k \notin \hat{K}_c^*$ and all n, it follows that for all $n > n'$ we have:

$$([b_{c,\hat{K}_c^*}^n]_+^T \ b_{c,L_c}^{n,T})^T = [G_{\hat{K}_c^*}^{ind}(x_c^n, \theta^n) \ H_{L_c}^{ind}(x_c^n)]^{-1} \cdot F_{c,J_c^+}^{ind}(x_c^n, \theta^n) \cdot [b_{c,J_c^+}^n]_+.$$

we conclude the proof of this step by invoking *Steps 1* and *5* to obtain

$$([b^n_{c,\hat{K}^*_c}]^T_+ \; b^{n,T}_{c,L_c})^T \to (\overset{+}{b}{}^T_{c,\hat{K}^*_c} \; b^{*,T}_{c,L_c})^T = [G^{ind}_{\hat{K}^*_c}(x^*_c,\theta^*) \; H^{ind}_{L_c}(x^*_c)]^{-1} \cdot F^{ind}_{c,J^+_c}(x^*_c,\theta^*) \cdot \overset{+}{b}_{c,J^+_c}.$$

For the last three steps we define the pseudo-inverse functions $[\cdot]^{-1}_+ : \mathbb{R}_+ \to \mathbb{R}_+$ and $[\cdot]^{1-1}_0 : [0,1] \to [0,1]$ where $[y]^{-1}_+ = \{x \in \mathbb{R}_+ : [x]_+ = y\}$ and $[y]^{1-1}_0 = \{x \in [0,1] : [x]^1_0 = y\}$. Note that these functions are continuous and that by definition $x = [[x_+]]^{-1}_+ - [[-x]_+]^{-1}_+$ for all $x \in \mathbb{R}$.

Step 11: *$b^n_{c,r} \to b^*_{c,r}$ for all c and all $r \in J^+_c \cup K_c \setminus (\mathring{J}_c \cup \mathring{K}_c)$.* By *Steps 5, 9, 10*, $b^n_{c,r} = [[b^n_{c,r}]_+]^{-1}_+ - [[-b^n_{c,r}]_+]^{-1}_+ \to [\overset{+}{b}_{c,r}]^{-1}_+ - [\bar{b}_{c,r}]^{-1}_+ = b^*_{c,r}$.

Step 12: *$p^n_c \to p^*_c$ for all s and all $c \notin \mathring{C}^p_s$.* By *Steps 2, 8*, $p^n_c = [[p^n_c]_+]^{-1}_+ - [[-p^n_c]_+]^{-1}_+ \to [\overset{+}{p}_c]^{-1}_+ - [\bar{p}_c]^{-1}_+ = p^*_c$.

Step 13: *$a^n_c \to a^*_c$ for all $c \notin \mathring{C}^a_s$.* By *Steps 3, 7, 11* and equation (3g) we have $[a^n_c - 1]_+ \to b^*_{c,i}$, $[a^n_c]^1_0 \to \overset{+}{a}_c$, and $[-a^n_c]_+ \to \bar{a}_c$. Thus, if $b^*_{c,i} > 0$ then $a^n_c \to a^*_c = 1 + [b^*_{c,i}]^{-1}_+$; if $\bar{a}_c > 0$, then $a^n_c \to a^*_c = -[\bar{a}_c]^{-1}_+$; and $a^n_c \to a^*_c = [\overset{+}{a}_c]^{1-1}_0$ otherwise.

To summarize, with *Steps 1,4,6,10,11,12,* and *13* we have shown that $e^n \to e^*$. Furthermore, $x^*_c \in \bar{X}^{\varepsilon_m}_c$ for all c (by *Step 1*) and $[-p^*_c]_+ \le \bar{u}(\theta^*) + \varepsilon_m$ for all $c \le S$ and $[-p^*_c]_+ \le 2(\bar{u}(\theta^*) + \varepsilon_m)$ for all $c > S$ (by *Step 8*). Since $\theta^* \in \Theta_m$ it follows that $(e^*, \theta^*) \in \bar{\Xi}_m$, and the proof is complete. □

Note that an immediate implication of Lemma 5 is that Θ_m is open for all m. We have thus paved the way for application of standard differential topology arguments on the mappings $\Phi^\nu_{m,\mu}$. Recall that y is a regular value of a C^1 function f if $Df(x)$ has full rank for all x such that $f(x) = y$. We show:

Lemma 6. *For all $m = 0, 1, \ldots, M$, all $\mu \in \mathcal{I}_m$, and all $\nu \in N(\mu)$, zero is a regular value of $\Phi^\nu_{m,\mu}$.*

Proof. Fix m, $\mu \in \mathcal{I}_m$, $\nu \in N(\mu)$, and $(e,\theta) \in \Xi_m$ such that $\Phi^\nu_{m,\mu}(e,\theta) = 0$. We wish to show that $\text{rank}(D\Phi^\nu_{m,\mu}(e,\theta)) = \#(\mu,\nu)$. We first show that e cannot take certain values using five Steps:

Step 1: *$b_{c,r} \neq 0$ for all c and all $r \in (J^-_c \cup K_c) \setminus (\mathring{J}_c \cup \mathring{K}_c)$.* Suppose instead that there exist c and $r \in (J^-_c \cup K_c) \setminus (\mathring{J}_c \cup \mathring{K}_c)$ such that $b_{c,r} = 0$ to get a contradiction. Observe that (e,θ) also satisfies $\Phi^0_{m,\mu}(e,\theta) = 0$. We construct $\mu' \in \mathcal{I}_{m+1}$ which is otherwise identical to μ except for $\mathring{J}'_c \cup \mathring{K}'_c = \mathring{J}_c \cup \mathring{K}_c \cup \{r\}$ and let $\nu' = (c,r)$. Note that $\Phi^{\nu'}_{m+1,\mu'}(e,\theta)$ is identical, up to row rearrangement, to $\begin{pmatrix} b_{c,r} \\ \Phi^0_{m,\mu}(e,\theta) \end{pmatrix} = 0$. we now have $(e,\theta) \in \Xi_m \subset \bar{\Xi}_{m+1}$, $\Phi^{\nu'}_{m+1,\mu'}(e,\theta) = 0$. But $\#(\mu',\nu') > \dim(E)$ because $\nu' \neq 0$ and $\theta \in \Theta^{\nu'}_{m+1,\mu'} \cap \Theta_m$, a contradiction.

Step 2: *$p_c \neq 0$ and $a_c \neq 1$ for all s and all $c \in C_s \setminus \mathring{C}_s$.* The proof is identical to *Step 1* this time genereting μ' by setting $\mathring{C}^{p\prime}_c = \mathring{C}^p_c \cup \{c\}$ (to show $p_c \neq 0$) or $\mathring{C}^{a\prime}_c = \mathring{C}^a_c \cup \{c\}$ (to show $a_c \neq 1$) and setting $\nu' = (s,c)$.

Step 3: *$a_c \neq 0$ for all s and all $c \in C_s$.* Assume there exist s, c, such that $c \in C_s$ and $a_c = 0$. Distinguish three cases, all of which lead to a contradiction. Case 1 ($c \in \mathring{C}^a_s$): then $a_c = 1$ from equation ($3\mathring{f}_a$). Case 2 ($c \in \mathring{C}_s^p \setminus \mathring{C}_s^a$): then $p_c = 0$ from equation ($3\mathring{f}_p$). We now create $\mu' \in \mathcal{I}_{m+1}$ otherwise identical to μ except for $\mathring{C}_s^{a\prime} = \mathring{C}_s^a \cup \{c\}$, and e' otherwise identical to e except for $a'_c = 1 \neq a_c = 0$. Since $[p_c]_+[a_c]_0^1 = [p_c]_+[a'_c]_0^1 = 0$, $[-a_c]_+ = [-a'_c]_+ = 0$, and $[a_c - 1]_+ = [a'_c - 1]_+ = 0$, we claim that $\Phi^0_{m+1,\mu'}(e', \theta) = 0$. Indeed, (e, θ) also satisfies $\Phi^0_{m,\mu}(e, \theta) = 0$ which differs from $\Phi^0_{m+1,\mu'}$ only with respect to the addition of equation ($3\mathring{f}_a$) for $c \in C_s$. But $\#(\mu', 0) > \dim(E)$, because $\dot{C}'_s \neq \emptyset$ and $\theta \in \Theta^0_{m+1,\mu'} \cap \Theta_m$, a contradiction. Case 3 ($c \in C_s \setminus \mathring{C}_s$): we now create $\mu' \in \mathcal{I}_{m+2}$ otherwise identical to μ except for $\mathring{C}_s^{a\prime} = \mathring{C}_s^a \cup \{c\}$ and $\mathring{C}_s^{p\prime} = \mathring{C}_s^p \cup \{c\}$. We also let e' be otherwise identical to e except for $a'_c = 1 \neq a_c = 0$, $p'_c = 0$, and p'_s such that $[p'_s]_+ = [p_s]_+ + [p_c]_+$ if $p_c > 0$ and $p'_s = p_s$ if $p_c \leq 0$. Note that in either case $[-p'_s]_+ = [-p_s]_+$, $[p'_s]_+ + [p'_c]_+ = [p_s]_+ + [p_c]_+$, $[p_c]_+[a_c]_0^1 = [p'_c]_+[a'_c]_0^1 = 0$, $[-a_c]_+ = [-a'_c]_+ = 0$, and $[a_c - 1]_+ = [a'_c - 1]_+ = 0$. Thus, $\Phi^0_{m+2,\mu'}(e', \theta) = 0$ since this system of equations differs from $\Phi^0_{m,\mu}(e, \theta) = 0$ in that equation ($3f$) is replaced with ($3\mathring{f}_a$) and ($3\mathring{f}_p$). But $\#(\mu', 0) > \dim(E)$, because $\dot{C}'_s \neq \emptyset$ and $\theta \in \Theta^0_{m+2,\mu'} \cap \Theta_m$, a contradiction.

Step 4: *For all s and all $c \in C_s \setminus \mathring{C}_s$, if $a_c \in (0, 1)$ then $U_i(e, c, \theta) \neq U_i(e, s, \theta)$.* Assume there exist s, c, such that $c \in C_s \setminus \mathring{C}_s$, $a_c \in (0, 1)$, and $U_i(e, c, \theta) = U_i(e, s, \theta)$ to get a contradiction. By virtue of equation ($3f$) we can distinguish two cases. Case 1 ($[-p_s]_+ = [-p_c]_+ > 0$): in this case e' that differs from e only at $a'_c = 0$ also satisfies $\Phi^\nu_{m,\mu}(e', \theta) = 0$, contradicting *Step 3*. Case 2 ($[-p_s]_+ = [-p_c]_+ = 0$): in this case construct e' which differs from e only with respect to coordinates a_c, p_c, p_s with the new coordinates taking values such that $a'_c = 1$, $[p'_c]_+ = [p_c]_+[a_c]_0^1$, and $[p'_s]_+ = [p_s]_+ + (1 - [a_c]_0^1)[p_c]_+$. We now have $\Phi^\nu_{m,\mu}(e', \theta) = 0$, a contradiction by *Step 2*.

Step 5: *If $\nu = (s, c)$ then $a_c > 0$.* By *Step 3*, $a_c \neq 0$ so assume $\nu = (s, c) \in N(\mu)$ and $a_c < 0$ to get a contradiction. Since $\nu \in N(\mu)$, $c \in \mathring{C}_s$. It cannot by that $c \in \mathring{C}_s^a$ as $a_c = 1 > 0$ from equation ($3\mathring{f}_a$), contradicting our assumption that $a_c < 0$. So $c \in \mathring{C}_s^p \setminus \mathring{C}_s^a$ and $p_c = 0$ from equation ($3\mathring{f}_p$). Then equation (3ν) becomes $[-p_s]_+ + [-a_c]_+ \neq 0$, a contradiction.

We can now show that $\operatorname{rank}(D\Phi^\nu_{m,\mu}(e, \theta)) = \#(\mu, \nu)$. First, if $\nu \neq 0$, let $\nu = (s, c')$ or $\nu = (c', r)$. Recalling that equation (3ν) appears only if $\nu \neq 0$, we compute

$$D\Phi^\nu_{m,\mu}(e, \theta) = \begin{pmatrix} \star & 0 & 0 & \Delta_\nu & \mathcal{Z}_\nu & \mathcal{G}_\nu & \star \\ \Omega_{c'} & 0 & 0 & \Delta_{c'} & \mathcal{Z}_{c'} & \mathcal{G}_{c'} & \star \\ 0 & \mathcal{D}\left((\Omega_c)_{c \neq c'}\right) & 0 & \Delta_c & \mathcal{Z}_c & \mathcal{G}_c & \star \\ 0 & 0 & \mathcal{D}([p_+]'_+) & 0 & 0 & 0 & \star \\ \star & \star & \star & I_{Z \cdot J} - \hat{\Delta} & \hat{\mathcal{Z}} & 0 & \star \end{pmatrix} \begin{matrix} (3\nu) \\ (3a)\text{-}(3g)_{c=c'} \\ (3a)\text{-}(3g)_{c \neq c'} \\ (3h) \\ (3i) \end{matrix},$$

$$\begin{matrix} y_{c'} & (y_c)_{c \neq c'} & p_+ & (\theta_z)_{z=1}^Z & V & \theta^0 & \text{other variables} \end{matrix}$$

where the variables with respect to which differentiation is performed are

$$y_c = \begin{cases} (x_c, b_c, p_c, a_c, \theta_c), & \text{if } c \in \dot{C}_{s_c}, \\ (x_c, b_c, a_c, \theta_c), & \text{if } c \in \mathring{C}_{s_c} \setminus \mathring{C}^p_{s_c} \text{ or } a_c > 0, \\ (x_c, b_c, p_c, \theta_c), & \text{if } c \in \mathring{C}_{s_c} \setminus \mathring{C}^a_{s_c} \text{ or } a_c < 0, \end{cases}$$

and p_+ is a $S \times 1$ vector that consists of one coordinate $p_c, c \in \bar{C}_s$, for each s with the property that $p_c > 0$ (such a coordinate exists by equation (3h)).[20] Stars ($\star$) indicate entries whose value does not affect rank, and $\mathcal{D}(\cdot)$ is a block diagonal matrix each diagonal block corresponding to one coordinate (possibly matrix-valued) of the input vector. $[p_+]'_+$ is a $S \times 1$ vector with coordinates the partial derivatives $\frac{\partial [p_c]_+}{\partial p_c}$, one for each of the S coordinates p_c of p_+. It follows that $\mathcal{D}([p_+]'_+)$ has full rank since $p_+ \gg 0$. By **(A3)** and the definition of $U_j(x_c, c, \theta)$, for all c, z_c and j, whenever parameter $\theta_j^{z_c}$ appears in one of the equations of system $\Phi^\nu_{m,\mu}(e,\theta) = 0$ in (3), V_j^z also appears in the same equation in the form

$$\star \cdot (f_j(x_c, c, \theta_c) + \theta_j^{z_c} + \delta_j^{z_c} V_j^{z_c}),$$

for some value $\star$. Thus, for all z, j we can multiply the column of matrix $D\Phi^\nu_{m,\mu}(e,\theta)$ that corresponds to θ_j^z with $-\delta_j^z$ and add the product to the column that corresponds to V_j^z, an operation that preserves rank and produces

$$D\Phi^\nu_{m,\mu}(e,\theta) \sim \begin{pmatrix} \omega_\nu & \star & 0 & 0 & 0 & \star \\ 0 & \Omega_{c'} & 0 & 0 & 0 & \star \\ \star & 0 & \mathcal{D}\left((\Omega_c)_{c\neq c'}\right) & 0 & 0 & \star \\ 0 & 0 & 0 & \mathcal{D}([p_+]'_+) & 0 & \star \\ \star & \star & \star & \star & I_{Z\cdot J} & \star \\ y_\nu & y_{c'} & (y_c)_{c\neq c'} & p_+ & V & \text{other variables} \end{pmatrix} \begin{matrix} (3\nu) \\ (3a)\text{-}(3g)_{c=c'} \\ (3a)\text{-}(3g)_{c\neq c'} \\ (3h) \\ (3i) \\ \end{matrix}.$$

Note that parameters θ^0, θ^z have now been omitted except (possibly) for the column that corresponds to variable

$$y_\nu = \begin{cases} \theta_k^0, & \text{if } \nu = (c', k), \\ \theta_j^{z_{c'}}, & \text{if } \nu = (c', j), \\ \theta_{i_s}^{z_{c'}}, & \text{if } \nu = (s, c'). \end{cases}$$

Now, in the relevant case, i.e., when $\nu \neq 0$, the scalar $\omega_\nu \neq 0$, since

$$\omega_\nu = \begin{cases} \frac{g_k(x_{c'}, \theta)}{\partial \theta_k^0} & \text{if } \nu = (c', k), \\ 1 & \text{if } \nu = (c', j), \\ [a_{c'}]_0^1 & \text{if } \nu = (s, c'), \end{cases}$$

by **(A3)** and *Step 4* which ensures that $[a_{c'}]_0^1 \neq 0$ when $\nu = (s, c')$. It then suffices to show that Ω_c has full rank for all c in order to establish that $D\Phi^\nu_{m,\mu}(e,\theta)$ has full rank. Ω_c takes one of the

[20] Note that whenever p_c appears in y_c, then either $p_c = 0$ (by equation ($3\mathring{f}_p$) when $c \in \mathring{C}^p_{s_c}$) or $p_c < 0$ (by equation (3f) when $a_c < 0$).

following forms:

$$\Omega_c = \begin{cases} \begin{pmatrix} A_c & 0 \\ \star & -\frac{\partial[a_c-1]_+}{\partial a_c} \end{pmatrix} \begin{matrix} (3a)-(3f) \\ (3g) \end{matrix}, & \text{if } a_c > 1, c \notin \mathring{C}_{s_c}, \\ \quad x_c, b_c, \theta_c \qquad a_c & \\ & \\ \begin{pmatrix} A_c & 0 & 0 \\ \star & B_c & 0 \\ \star & 0 & 1 \end{pmatrix} \begin{matrix} (3a)-(3e) \\ (3\mathring{f}_p)-(3f) \\ (3g) \end{matrix}, & \text{otherwise.} \\ \quad x_c, b_{c,-i}, \theta_c \quad a_c \text{ and/or } p_c \quad b_{c,i} & \end{cases}$$

Since $\frac{\partial[a_c-1]_+}{\partial a_c} > 0$ when $a_c > 1$, in order to show that Ω_c has full rank, we need to show that A_c and B_c (when applicable) have full rank. First, since we have ruled out the case $a_c \in \{0,1\}$ for $c \notin \mathring{C}_{s_c}$ by *Steps 1-2*, the possible values of B_c are

$$B_c = \begin{cases} I_2, & \text{if } c \in \dot{C}_{s_c}, \\ 1, & \text{if } c \in \mathring{C}_{s_c} \setminus \dot{C}_{s_c}, \\ \dfrac{\partial[a_c]_0^1}{\partial a_c}(U_i(e,c,\theta) - U_i(e,s_c,\theta)), & \text{if } a_c \in (0,1), c \notin \mathring{C}_{s_c}, \\ -[-p_c]'_+, & \text{if } a_c < 0, c \notin \mathring{C}_{s_c}. \end{cases}$$

Since $p_c < 0$ when $a_c < 0$ (by equation (3f)), $[-p_c]'_+ \neq 0$. Also, $\frac{\partial[a_c]_0^1}{\partial a_c}(U_i(e,c,\theta) - U_i(e,s_c,\theta)) \neq 0$ when $a_c \in (0,1)$, by *Step 3*. It follows that B_c has full rank, so that Ω_c has full rank if A'_c does, where (after rearranging rows and columns and performing elementary operations[21] that preserve rank and making use of *Step 1* that ensures that $b_{c,r} \neq 0$ for all $r \notin \mathring{J}_c^- \cup \mathring{K}_c$) $A_c \sim A'_c$ and

$$A'_c = \begin{pmatrix} x_c & b_{c,\hat{J}_c^+} & b_{c,\hat{K}_c} & b_{c,L_c} & b_{c,J_c^-\setminus\hat{J}_c^+\cup K_c\setminus\hat{K}_c} & \theta_c \\ \Gamma_c & F_{\hat{J}_c^+}(x_c,c,\theta) & G_{\hat{K}_c}(x_c,\theta) & H_{L_c}(x_c) & 0 & D_{\theta_c}F_{\hat{J}_c^+}(x_c,c,\theta) \\ F_{\hat{J}_c^+}(x_c,c,\theta)^T & 0 & 0 & 0 & 0 & D_{\theta_c}f_{\hat{J}_c^+}(x_c,c,\theta) \\ G_{\hat{K}_c}(x_c,\theta)^T & 0 & 0 & 0 & 0 & 0 \\ H_{L_c}(x_c)^T & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{\#(J_c^-\setminus\hat{J}_c^+\cup K_c\setminus\hat{K}_c)} & 0 \\ 0 & \mathbf{1}^T & 0 & 0 & 0 & 0 \end{pmatrix},$$

with $\Gamma_c = D_{x_c}(F_{\hat{J}_c^+}(x_c,c,\theta)\cdot[b_{c,\hat{J}_c^+}]_+ + G_{\hat{K}_c}(x_c,\theta)\cdot[b_{c,\hat{K}_c}]_+ + H_{L_c}(x_c)\cdot b_{c,L_c})$, and $\hat{J}_c^+ = \{j \in J_c^+ \mid b_{c,j} > 0\}$, $\hat{K}_c = \{k \in K_c \mid b_{c,k} > 0\} \subseteq K_c(x_c,\theta)$. We now show that $\lambda^T = \begin{pmatrix} \lambda_1^T & \lambda_2^T & \lambda_3^T & \lambda_4^T & \lambda_5^T & \lambda_6^T \end{pmatrix}$ can satisfy $\lambda^T \cdot A'_c = 0$ if and only if $\lambda = 0$. Indeed, if $\lambda^T \cdot A'_c = 0$ we immediately conclude that

[21] In particular, dividing columns corresponding to $b_{c,r} > 0$ by $[b_{c,r}]_+$ and those corresponding to $b_{c,r} < 0$ by $-[-b_{c,r}]_+$.

$\lambda_5 = 0$. Furthermore,

$$\begin{aligned}
\lambda_1^T \cdot \Gamma_c + \lambda_2^T \cdot F_{\hat{J}_c^+}(x_c, c, \theta)^T + \lambda_3^T \cdot G_{\hat{K}_c}(x_c, \theta)^T + \lambda_4^T \cdot H_{L_c}(x_c)^T &= 0 \\
\lambda_1^T \cdot F_{\hat{J}_c^+}(x_c, c, \theta) + \lambda_6 \cdot \mathbf{1}^T &= 0 \\
\lambda_1^T \cdot G_{\hat{K}_c}(x_c, \theta) &= 0 \\
\lambda_1^T \cdot H_{L_c}(x_c) &= 0 \\
\lambda_1^T \cdot D_{\theta_c} F_{\hat{J}_c^+}(x_c, c, \theta) + \lambda_2^T \cdot D_{\theta_c} f_{\hat{J}_c^+}(x_c, c, \theta) &= 0.
\end{aligned}$$

Post-multiplying the second equation with $[b_{c,\hat{J}_c^+}]_+$, noting that $\mathbf{1}^T \cdot [b_{c,\hat{J}_c^+}]_+ = 1$ by equation (3e), and substituting from (3a) we obtain

$$-\lambda_1^T \cdot (G_{\hat{K}_c}(x_c, \theta) \cdot [b_{c,\hat{K}_c}]_+ + H_{L_c}(x_c) \cdot b_{c,L_c}) + \lambda_6 = 0,$$

which implies $\lambda_6 = 0$ and $\lambda_1^T \cdot F_{\hat{J}_c^+}(x_c, c, \theta) = 0$ (by the third and fourth equations). Post-multiplying the first equation by λ_1 we now obtain $\lambda_1^T \cdot \Gamma_c \cdot \lambda_1 = 0$. But by **(A6)**, $\lambda_1^T \cdot \Gamma_c \cdot \lambda_1 < 0$ for all non-zero λ_1 such that $\lambda_1^T \cdot F_{\hat{J}_c^+}(x_c, c, \theta) = 0, \lambda_1^T \cdot G_{\hat{K}_c}(x_c, \theta) = 0, \lambda_1^T \cdot H_{L_c}(x_c) = 0$, so we conclude that $\lambda_1 = 0$. The first and last equations now become

$$\begin{aligned}
\lambda_2^T \cdot F_{\hat{J}_c^+}(x_c, c, \theta)^T + \lambda_3^T \cdot G_{\hat{K}_c}(x_c, \theta)^T + \lambda_4^T \cdot H_{L_c}(x_c)^T &= 0 \\
\lambda_2^T \cdot D_{\theta_c} f_{\hat{J}_c^+}(x_c, c, \theta) &= 0.
\end{aligned}$$

we now claim that there does not exist $r \in \hat{J}_c^+ \cup \hat{K}_c$ and $b'_{c,\hat{J}_c^+ \cup \hat{K}_c}$ with $b'_{c,r} = 0$ such that equations (3a) and (3e) (and (3g) if applicable, i.e., if $a_c > 1$) are satisfied when we replace $b_{c,\hat{J}_c^+ \cup \hat{K}_c^+}$ with $b'_{c,\hat{J}_c^+ \cup \hat{K}_c}$. For suppose such an r and $b'_{c,\hat{J}_c^+ \cup \hat{K}_c}$ exist. Then, if $r \neq i$ the resulting vector e' which is otherwise identical to e also solves $\Phi^\nu_{m,\mu}(e', \theta) = 0$. But we have $r \notin \mathring{J}_c^- \cup \mathring{K}_c$ so $b'_{c,r} = 0$ contradicts *Step 1*. If, on the other hand, $r = i$ (and $a_c > 1$ so that $b_{c,i} > 0$) then once more e' otherwise identical to e except for $b'_{c,\hat{J}_c^+ \cup \hat{K}_c}$ replacing $b_{c,\hat{J}_c^+ \cup \hat{K}_c}$ and $a'_c = 1$ so that $[a'_c - 1]_+ = b'_{c,i} = 0$ also solves $\Phi^\nu_{m,\mu}(e', \theta) = 0$. This time, $c \notin \mathring{C}_{s_c}$ and the existence of e' contradicts *Step 2*. We then conclude that $(\lambda_2, \lambda_3, \lambda_4) = 0$ by Theorem 12. Thus, $\lambda^T \cdot A'_c = 0$ implies $\lambda = 0$, hence A_c and $\Omega_c, c > S$, all have full rank, completing the proof. □

Next, we repeatedly apply Sard's Theorem to conclude:

Lemma 7. *Θ_m is an open, full measure subset of Θ for all $m = 0, 1, \ldots, M$.*

Proof. Obvious for $m = M$ since $\Theta_M = \Theta$. We thus proceed inductively, fixing $m = 1, \ldots, M$ and assuming Θ_m is an open, full measure subset of Θ. Then it suffices to show that Θ_{m-1} also is an open, full measure subset of Θ. For all $\mu \in \mathcal{I}_m$, and all $\nu \in N(\mu)$, define the manifold

$$\mathcal{M}^\nu_{m,\mu} = \{(e, \theta) \in \Xi_m \mid \Phi^\nu_{m,\mu}(e, \theta) = 0\}.$$

Step 1: *For all $\mu \in \mathcal{I}_m$, and all $\nu \in N(\mu)$, $\mathcal{M}^\nu_{m,\mu}$ is a $(\dim(E) + \dim(\Theta) - \#(\mu, \nu))$-dimensional C^1 manifold (or it is empty).* By Lemma 5 and the inductive hypothesis Ξ_m is a non-empty open set, hence a smooth manifold of dimension $\dim(E)+\dim(\Theta)$. By Lemma 6 zero is a regular value of $\Phi^{\nu}_{m,\mu}$. Since $\Phi^{\nu}_{m,\mu}$ is C^1 the result follows from the Regular value Theorem (Villanacci et al. (2002), Theorem 9, page 84).

Step 2: *For all $\mu \in \mathcal{I}_m$, and all $\nu \in N(\mu)$ such that $\#(\mu,\nu) > \dim(E)$, $\Theta^{\nu}_{m,\mu}$ has measure zero in Θ.* By Step 1, $\mathcal{M}^{\nu}_{m,\mu}$ is a C^1 manifold. Let $pr_{|\mathcal{M}^{\nu}_{m,\mu}}$ denote the restriction of the natural projection $pr : E\times\Theta \to \Theta$ on $\mathcal{M}^{\nu}_{m,\mu}$. Since $\dim(\Theta) > \dim(\mathcal{M}^{\nu}_{m,\mu})$, the set of critical values of $pr_{|\mathcal{M}^{\nu}_{m,\mu}}$ is the set of all points in the image set $pr(\mathcal{M}^{\nu}_{m,\mu})$. Furthermore, since $\#(\mu,\nu) > \dim(\Theta)$

$$\Theta^{\nu}_{m,\mu} = \{\theta \in \Theta_m \mid \exists (e,\theta) \in \bar{\Xi}_m \text{ such that } \Phi^{\nu}_{m,\mu}(e,\theta) = 0\} = pr(\mathcal{M}^{\nu}_{m,\mu} \cap \bar{\Xi}_m) \subseteq pr(\mathcal{M}^{\nu}_{m,\mu}).$$

By Sard's Theorem (Villanacci et al. (2002), Theorem 23, page 150) we conclude that the set of critical values, $pr(\mathcal{M}^{\nu}_{m,\mu})$, of $pr_{|\mathcal{M}^{\nu}_{m,\mu}}$ has measure zero in Θ. Then $\Theta^{\nu}_{m,\mu}$ is a measure zero subset of Θ, as we wished to show.

It follows from Lemma 5, the definition of Θ_{m-1}, and *Step 2*, that Θ_{m-1} is an open full measure subset of Θ for all $m = 1,\dots,M$. □

With Lemmas 4, 5, 6, and 7 we are ready to prove:

Lemma 3 (restated)**.** *There exists an open, full measure subset of Θ, $\Theta^* \subseteq \Theta$, such that for all $\theta \in \Theta^*$ all equilibria are regular.*

Proof. Define the mapping $\Phi^* : E\times\Theta_0 \to \mathbb{R}^{\dim(E)}$ as a restriction of Φ, that is, $\Phi^*(e,\theta) = \Phi(e,\theta)$ for all $(e,\theta) \in E\times\Theta_0$. By Lemma 4, $\Phi^{*-1}(0) \subseteq \bar{\Xi}_0 \subset \Xi_0$. Furthermore, $\mathcal{I}_0 = \{\mu \in \mathcal{I} \mid \mathring{C}^p_s = \mathring{C}^a_s = \emptyset$ for all s, $\mathring{J}_c = \mathring{K}_c = \emptyset$ for all $c\} = \{\mu_0\}$, so that $\Phi^0_{0,\mu_0}(e,\theta) = \Phi^*(e,\theta)$ for all $(e,\theta) \in \Xi_0$, the domain of Φ^0_{0,μ_0}. Then zero is a regular value of Φ^* since it is a regular value of Φ^0_{0,μ_0}, by Lemma 6. We then apply the Transversality Theorem (Villanacci et al. (2002), Theorem 26, page 151) on Φ^* to conclude that there exists a full measure subset Θ^* of Θ_0 such that $\text{rank}(D_e\Phi(e,\theta)) = \dim(E)$ for all $\theta \in \Theta^*$ and all e such that $\Phi(e,\theta) = 0$. Since Φ^* and Φ^0_{0,μ_0} coincide on Ξ_0 and all solutions to $\Phi^*(e,\theta) = 0$ lie in a subset of Ξ_0, we set $\Theta^* = \Theta_0 \setminus \Theta^0_{0,\mu_0}$. Since Θ^0_{0,μ_0} is closed in Θ_0 by Lemma 5, we conclude that Θ^* is an open full measure subset of Θ. □

For the remainder, add discount factors as an argument in function Φ, that is $\Phi : E\times\Theta\times[0,1)^{ZJ} \to \mathbb{R}^{\dim(E)}$ is the left-hand-side of equations (1) with arguments (e,θ,δ) and $\delta = (\delta^z_j)_{j,z}$. By Lemma 3, for all δ there exists an open, full measure set $\Theta^*_\delta \subset \Theta$ such that for $\theta \in \Theta^*_\delta$ and all e such that $\Phi(e,\theta,\delta) = 0$, $\text{rank}(D_e\Phi(e,\theta,\delta)) = \dim(E)$.

The next Lemma states that when parameters θ are drawn from the set Θ^*_0, the game with zero discount factors has a unique equilibrium.

Lemma 8. *For all $\theta_0 \in \Theta^*_0$ there exists a unique $e^* \in E$ such that $\Phi(e^*,\theta_0,0) = 0$.*

Proof. Fix $\theta_0 \in \Theta^*_0$ and note that $\bar{A}(c;\sigma) = \bar{A}(c) = \{x \in X_c(\theta_0) \mid f_{J^-}(x,c,\theta_0) \geq f_{J^-}(x_{s_c},s_c,\theta_0)\}$ and $A(c;\sigma) = A(c) = \{x \in X_c(\theta_0) \mid f_{J^-}(x,c,\theta_0) > f_{J^-}(x_{s_c},s_c,\theta_0)\}$ for all c and all σ, because $\delta = 0$. Thus, the optimization program $\max_x\{f_i(x,c,\theta_0) \mid x \in \bar{A}(c)\}$ coincides with $P(c;\sigma)$ for all c,σ so that if $\bar{A}(c) \neq \emptyset$ it admits a unique solution, say x^*_c, by Lemma 1. Let $C^*_s = \{c \in C_s \mid \bar{A}(c) \neq \emptyset\}$

and select $c_s^* \in \arg\max_{c \in C_s^*}\{f(x_c^*, c, \theta_0)\}$ for all s. Since $\delta = 0$ it follows that there exists an equilibrium σ^* such that $\pi_s^*(\{(x_{c_s^*}^*, c_s^*)\}) = 1$ for all s and $\alpha(x, c; \sigma^*) = 1$ for all c and all $x \in \bar{A}(c)$. By Theorem 2 there exists e^* such that $\sigma = \hat{\sigma}(e^*)$ and $\Phi(e^*, \theta_0, 0) = 0$. Since $\theta_0 \in \Theta_0^*$, $D_e\Phi(e, \theta_0, 0)$ has full rank for all e that solve $\Phi(e, \theta_0, 0) = 0$. Thus, by Lemma 2, specifically parts 1 and 2, $p_c^* \neq 0$ and $a_c^* > 1$ or $a_c^* < 0$ for all c. Then $p_c^* < 0$ for all $c \neq c_s^*$ by part 1 of Lemma 2, and from equation (1f) we conclude that $\arg\max_{c \in C_s^*}\{f(x_c^*, c, \theta_0)\} = \{c_s^*\}$. Furthermore, $A(c_s^*) \neq \emptyset$, for otherwise part 3 of Lemma 1 implies there is $e' \neq e^*$ identical in all coordinates except $b'_{c_i^*}$ and $a'_{c_s^*}$ such that $a'_{c_s^*} = 1$ and $b'_{c_s^*,i} = 0$, hence $\sigma = \hat{\sigma}(e') = \hat{\sigma}(e^*)$, contradicting Lemma 2. It follows that at each s, $x_{c_s^*}^*$ is the unique optimal proposal that is accepted with probability one. Then equilibrium σ is unique and e^* is the unique solution of $\Phi(e, \theta_0, 0) = 0$ by Lemma 2. □

Fix δ and $\theta_0 \in \Theta_0^*$ and define the following *homotopy* function $\Phi_H : E \times [0,1] \times \Theta \to \mathbb{R}^{\dim(E)}$ as

$$\Phi_H(e, t, \theta) = \Phi(e, (1-t)\theta_0 + t\theta, t\delta).$$

By essentially repeating the arguments of Lemma 5 we establish:

Lemma 9. *The set $\{(e,t) \in E \times [0,1] \mid \Phi_H(e,t,\theta) = 0\}$ is compact for all $\theta \in \Theta$.*

Proof. Consider a sequence $(e^n, t^n) \in E \times [0,1]$ such that $\Phi_H(e^n, t^n, \theta^*) = 0$ for all n. It suffices to show, going to a subsequence if necessary, that $(e^n, t^n) \to (e^*, t^*) \in E \times [0,1]$. Then $\Phi_H(e^*, t^*, \theta^*) = 0$ by continuity of Φ_H. With the exception of new *Step 0* and *Step 8*, the argument proceeds as in Lemma 5 (with $\mathring{J}_c = \mathring{K}_c = \mathring{C}_s^a = \mathring{C}_s^p = \emptyset$ for all c, s).

Step 0: *There exists a subsequence such that $t^n \to t^*$.* Since $t^n \in [0,1]$, a compact set.

Step 1: *There exists a subsequence such that $x_c^n \to x_c^*$ for all c.* Since $x_c^n \in X_c$, a compact set, for all c.

Step 2 to 5 and 7 (Step 6 does not apply) can be repeated as in Lemma 5 with $\theta^* = (1-t^*)\theta_0 + t^*\theta$ and δ replaced by $t^*\delta$. We continue with:

Step 8: *There exists a subsequence such that $[-p_c^n]_+ \to \bar{p}_c$ for all c.* Distinguish two cases. If at *Step 2* $\overset{+}{p}_c > 0$, then $[-p_c^n]_+ \to \bar{p}_c = 0$. Next consider cases such that $\overset{+}{p}_c = 0$. We will first show that $[-p_s^n]_+ \to \bar{p}_s$ for all s such that $\overset{+}{p}_s = 0$. Indeed, by *Step 2* there exists $c \in C_s \setminus \{s\}$ such that $\overset{+}{p}_c > 0$, so that $[-p_c^n]_+ \to 0$. Applying that fact and *Steps 0-4* in equation (1f) corresponding to c, observing that by that same equation it cannot be that $\bar{a}_c > 0$, we conclude that

$$[-p_s^n]_+ \to \bar{p}_s = \overset{+}{a}_c((f_j(x_c^*, c, \theta^*) + t^*\delta_j(z_c)V_{z_c,j}^*) - (f_j(x_s, s, \theta^*) + t^*\delta_j(z_s)V_{z_s,j}^*)).$$

Having established that $[-p_s^n]_+ \to \bar{p}_s$ for all s, we now turn to $[-p_c^n]_+$ such that $c \neq s$ and $\overset{+}{p}_c = 0$. By equation (1f) and *Steps 0-4, 7,* we conclude that

$$[-p_c^n]_+ \to \bar{p}_c = \bar{p}_s - \overset{+}{a}_c((f_j(x_c^*, c, \theta^*) + t^*\delta_j(z_c)V_{z_c,j}^*) - (f_j(x_s, s, \theta^*) + t^*\delta_j(z_s)V_{z_s,j}^*)) + \bar{a}_c.$$

The argument concludes as in Steps 9 to 13 of Lemma 5. □

In what follows, the notation $\#\phi^{-1}(0)$ stands for the cardinality of the solution set $\{x|\phi(x) = 0\}$, and for integer $n \geq 0$, $[n]_{\bmod 2}$ is one if n is odd and zero if n is even.

Theorem 3 (restated)**.** *An equilibrium exists for all $\theta \in \Theta$.*

Proof. Φ_H is a C^1 homotopy between $\Phi_0(e) = \Phi(e, \theta_0, 0)$ and $\Phi_1(e) = \Phi(e, \theta, \delta)$. Since $\theta_0 \in \Theta_0^*$, $\deg(\Phi_0, E, 0) = [\#\Phi_0^{-1}(0)]_{\bmod 2} = 1$, by Lemma 3 and Lemma 8 (see, e.g., Proposition 56, page 198, Villanacci et al. (2002)). Existence of solutions for $\Phi_1(e) = 0$ now follows by the smoothness of E and Lemma 9 (see, e.g., Theorem 57, page 199, Villanacci et al. (2002)). □

Theorem 4 (restated)**.** *There exists an open, full measure subset of Θ, $\Theta^* \subset \Theta$, such that for all $\theta \in \Theta^*$ the number of equilibria is odd and every equilibrium is regular.*

Proof. Let $\theta_0 \in \Theta_0^*$ and $\theta \in \Theta_\delta^*$. As in Theorem 3, Φ_H is a C^1 homotopy between $\Phi_0(e) = \Phi(e, \theta_0, 0)$ and $\Phi_1(e) = \Phi(e, \theta, \delta)$. Now, $\deg(\Phi_0, E, 0) = \deg(\Phi_1, E, 0) = [\#\Phi_1^{-1}(0)]_{\bmod 2} = 1$, by Lemma 9, Lemma 3, and Lemma 8. It follows that $\Phi(e, \theta, \delta)$ has an odd number of solutions for all $\theta \in \Theta_\delta^*$, an open, full measure subset of Θ by Lemma 3. By Theorem 2 and Lemma 2, these solutions correspond to distinct equilibria σ. □

APPENDIX B

The following Theorems are used in Appendix A:

Theorem 11 (Motzkin's Transposition Theorem)**.** *Let F, G, and H be matrices with $F \neq 0$. Either there exist $\beta_f, \beta_g, \beta_h$, with $\beta_f, \beta_g \geq 0, \beta_f \neq 0$, such that $F \cdot \beta_f + G \cdot \beta_g + H \cdot \beta_h = 0$, or there exists λ such that $\lambda^T \cdot F > 0$, $\lambda^T \cdot G \geq 0$, and $\lambda^T \cdot H = 0$.*

Theorem 12 (Caratheodory)**.** *Suppose $F_{\hat{J}}(x, c, \theta) \cdot \beta_{\hat{J}} + G_{\hat{K}}(x, \theta) \cdot \beta_{\hat{K}} + H_{L_c}(x) \cdot \beta_{L_c} = 0$ for some $c, x \in Y_c$, $\hat{J} \subseteq J_c^+, \hat{K} \subseteq K_c(x, \theta)$, $\beta_{\hat{J}}, \beta_{\hat{K}} \geq 0$, $\mathbf{1}^T \cdot \beta_{\hat{J}} = 1$. If there does not exist $r \in \hat{J} \cup \hat{K}$ and $\beta'_{\hat{J} \cup \hat{K} \cup L_c}$ such that $\beta'_{\hat{J} \cup \hat{K}} \geq 0$, $\beta'_r = 0$, $\mathbf{1}^T \cdot \beta'_{\hat{J}} = 1$, and $F_{\hat{J}}(x, c, \theta) \cdot \beta'_{\hat{J}} + G_{\hat{K}}(x, \theta) \cdot \beta'_{\hat{K}} + H_{L_c}(x) \cdot \beta'_{L_c} = 0$ then $\begin{pmatrix} F_{\hat{J}}(x, c, \theta) & G_{\hat{K}}(x, \theta) & H_{L_c}(x) \\ D_{\theta_c} f_{\hat{J}}(x, c, \theta)^T & 0 & 0 \end{pmatrix}$ has full column rank.*

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