

TURNOUT AND POWER SHARING

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ABSTRACT. Differences in electoral rules and/or legislative, executive or legal institutions across countries induce different mappings from election outcomes to distribution of power. We are interested in how such different mappings from election outcomes to distribution of power affect voters' participation in a democracy. Assuming heterogeneity in the cost of voting, the effect of such institutional differences on turnout depends on the expected closeness of the election: when two parties are expected to have similar support, turnout is higher the closer the system is to a winner take all one; the result is the opposite when one party has a larger expected base. We compare size effect and underdog effect under different systems. The results from the rational voters' model are robust to the introduction of parties' mobilization strategies. Turnout is also shown to increase with the number of parties in the proportional system.

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1. INTRODUCTION

Voters' participation is an essential component of democracy. Yet the positive analysis of turnout is still far from established and many questions remain. Is it possible to characterize the influence of institutional systems on turnout? In particular, does turnout depend in any identifiable way on the type of democratic regime, the electoral rules, legislative organization rules, or the degree of separation of powers? Our idea to make headway on this topic is as follows: all such important features of different institutional systems have an impact on the mapping from election outcomes (vote shares) to the relative weight of different parties in decision making (power shares); hence if we can fully characterize the dependence of turnout on that vote-shares-to-power-shares mapping, the role of institutional variations can be assessed through that intermediate step.¹

In other words, the objective of identifying some general way in which institutions affect mobilization efforts by parties and voters' incentives to vote can be attained, not by analyzing institutions one by one, but rather focusing on the mapping from vote shares to power shares as summary variable in explaining turnout. The role of individual institutions could be evaluated by referring to their expected impact on that abstract mapping. Since the degree of proportionality of influence on policy determination power given electoral outcomes is the key variable for our analysis, we will often refer to this reduced form mapping simply as the "institutional system", ranging from the pure proportional system to the winner-take-all one.

The results will depend crucially also on the interaction with another key parameter, namely the "expected closeness" of an election. Before explaining how the vote-share to power-share mapping interacts with the expected closeness of an election to determine turnout, we need to highlight the main modeling choices that we make in the paper.

We assume that voters' preferences over the set of alternatives (candidates or parties or coalitions of parties) are given and common knowledge, so that the only relevant decision by voters is whether to go to vote or not. Each voter is described in a two-dimensional type space,

¹The relative power of the majority party for a given election outcome varies with the degree of separation of powers, the organization of chambers, the assignment of committee chairmanships and institutional rules on agenda setting, allocation of veto powers, and obviously electoral rules. See Lijphart (1999) and Powell (2000) for a comprehensive analysis of the impact of political institutions on the degree of proportionality of influence. Electoral rules determine the mapping from vote shares to seat shares in a legislature, whereas the other institutions determine the subsequent mapping from seat shares to power shares across parties.

i.e. her preferred party and her cost of voting. Costs of voting are heterogeneous and the cost benefit analysis for the decision to vote depends therefore on individual as well as institutional characteristics.

In our rational voter model the distribution of voting costs is given, but the benefit of voting is endogenous to the institutional system. In a fully proportional system the expected marginal benefit of an individual vote is proportional to the marginal change in the vote share determined by the extra vote, whereas in a winner take all system the marginal benefit of a vote is proportional to the probability of that vote being pivotal. Both such marginal benefits obviously decrease as the number of voters increases. The comparison of turnout across systems will depend on the speed with which a larger electorate reduces the benefit of voting, i.e. the magnitude of the “size effect” for different ex ante evaluations of the relative strength of parties.

Beside the necessary comparison of the size effects across systems, we also study the “underdog effect”, i.e. the impact of a system and all the other parameters on the relative participation of the supporters of an underdog party vis a vis the participation of the favorite party supporters. In contrast with the case of homogeneous cost of voting, the underdog effect is characterized by a partial compensation, in the sense that the supporters of the underdog parties turn out in higher percentage but not enough to make the election become a coin toss.² This partial underdog effect varies with the degree of proportionality of influence induced by the institutional system, but is always present. Under some conditions on the distribution of voting costs in the population, the underdog effect is greater in a proportional influence system than in a winner-take-all one.

The key comparative result that we obtain comes from the difference in terms of size effect: in a proportional system we show that the benefit of voting decreases proportionally to $1/N$ when N , the expected size of the electorate, increases; on the other hand, in a winner-take-all system such a speed is slower when the election is expected to be a tie and much faster otherwise. This fact determines the main conclusion, namely that turnout is higher in a proportional system when the election has a clear favorite party while a winner-take-all system induces higher turnout otherwise.

Even though we conduct the bulk of the analysis for the case of two parties, we show the robustness of all comparisons to changes in the

²This contrasts with the result by Goree and Grosser (2007), in which there is full compensation and the election is a toss-up. Full compensation occurs if the cost of voting is the same for every agent as they assume.

number of parties: in a proportional system the size effect does not depend on the distribution of ex ante support of parties, and we show that the way turnout depends on the size of the electorate does not change with the number of parties either. Hence the comparison with the winner take all system is also unaffected by the number of parties. Finally, we show that in a proportional system turnout increases as the number of parties increases.

For robustness purposes, we also study the same questions with a model with opposite characteristics, i.e., fixing the individual benefit of voting and making parties choose mobilization strategies that affect the distribution of costs of voting. Even in that model we confirm that for symmetric priors turnout is higher with a winner take all system, and if the spread between favorite and underdog is sufficiently large it is higher in a system with full proportionality of influence.

From the modeling standpoint, Myerson (1998 and 2000) are the most important papers for us, since they supply many simplification results by viewing the size of the electorate as a Poisson random variable. This specification simplifies the analysis and the computations. In a recent working paper Krishna and Morgan (2009) use the same Poisson population uncertainty in a Condorcet common values model with costly voting, allowing for abstention and endogenous turnout. They show that in a large electorate there is a unique outcome, in which voters that decide to turnout to the polls vote sincerely and the turnout in favor of each candidate is such that the correct candidate is elected with probability one.

The paper that inspired our choice of conceptual categories for the study of turnout is Levine and Palfrey (2007), who studied the behavioral phenomena identified as “size effect and underdog effect” in a winner take all system experiment. Our paper generalizes the theoretical analysis and allows to compare such effects across systems.³

The model in the mobilization robustness section has features in common with Shachar and Nalebuff (1999), one of the seminal papers on mobilization efforts. Other relevant works will be cited in the body of the paper, and some others in the concluding remarks.

The paper is organized as follows. Section 2 contains the complete analysis of a rational voter model of turnout, comparing the properties of proportional system and winner-take-all system. Section 3 contains

³Some of our comparative results, for example that the size effect is stronger in a proportional system only when the election is expected to be close, but much stronger in a winner-take-all system when the election has a clear favorite, could be testable in an environment similar to their experimental design.

the analysis of the mobilization model, where even intermediate proportionality levels can be considered, and where we confirm the robustness of our main comparative results across modeling choices. Section 4 will offer some concluding remarks and describe potential paths of future research. All proofs are in the Appendix.

2. RATIONAL VOTER TURNOUT

Consider two parties, A and B, competing for power. Citizens have exogenous political preferences for one or the other, and we denote by q the fraction of citizens who prefer party A (thus a fraction $1-q$ prefer party B's policies). The indirect utility for a citizen of preference type i , $i = A, B$, is increasing in the share of power that party i has. For normalization purposes, we let the utility from "full power to party i " equal 1 for type i citizens and 0 for the remaining citizens.⁴

Beside partisan preferences, the second dimension along which citizens differ from one another is their cost of voting: each citizen's cost of voting is drawn from a distribution with cdf F . The cost of voting and the partisan preferences are two independent dimensions that determine the type of a voter.

For any vote share x obtained by party A, an institutional system γ determines power shares $B_\gamma^A(x) \in [0, 1]$ and $B_\gamma^B(x) = 1 - B_\gamma^A(x)$. Given the above normalization, these are the reduced form "benefit" components of parties' (respectively, voters') utility functions that will determine the incentives to, campaign (respectively, vote) in an institutional system. In this section we study the base model in which parties do not campaign nor attempt to mobilize voters, hence turnout depends exclusively on voters' comparison between the policy benefits of voting for the preferred candidate and the opportunity costs of voting.⁵

In terms of the size of the electorate, we find it convenient to assume that the population is finite but uncertain. There are n citizens who are able to vote at any given time, but such a number is uncertain and distributed as a Poisson distribution with mean N :

$$n \sim \frac{e^{-N} (N)^n}{n!}$$

Most statements in the paper are made for a large enough population, namely they are true for every N above a given \bar{N} .

⁴This normalization will allow us to match party utility and voters's utilities in a simple way under all the institutional systems that will be considered.

⁵In section 3 we will show that the comparative results are very similar when parties' mobilization strategies are considered.

Citizens have to choose to vote for party A, party B, or abstain. If a share α of A types vote for A and a share β of B types vote for B, the expected turnout T is

$$T = q\alpha + (1 - q)\beta$$

To analyze the cases in which $q \neq 1/2$, without loss of generality we often assume that $q < 1/2$, so that the A party is the underdog party (with smaller ex-ante support) and the B party is the leader party (with larger ex-ante support).

We look for a Bayesian equilibrium in which all voters of type A with a cost below a threshold c_α vote for type A and voters of type B with a cost below c_β vote for B. So type A citizens vote for A with chance $\alpha = F(c_\alpha)$ and type B citizens vote for B with chance $\beta = F(c_\beta)$.

In any equilibrium strategy profile (α, β) , the expected marginal benefit of voting B_γ must be equal to the cutoff cost of voting (indifference condition for the citizen with the highest cost among the equilibrium voters). Hence the equilibrium conditions can be written as

$$B_\gamma^A(\alpha, \beta) = F^{-1}(\alpha), \quad B_\gamma^B(\alpha, \beta) = F^{-1}(\beta)$$

We compare two systems: a winner-take-all system ($\gamma = M$) and a proportional system ($\gamma = P$).⁶

2.1. Winner take all system ($\gamma = M$). In the M system the expected marginal benefit of voting B_M^A is the chance of being pivotal for a type A citizen, namely $B_M^A =$

$$\sum_{k=0}^{\infty} \left(\frac{e^{-qN\alpha} (Nq\alpha)^k}{k!} \right) \left(\frac{e^{-(1-q)N\beta} ((1-q)N\beta)^k}{k!} \right) \frac{1}{2} \left(1 + \frac{(1-q)N\beta}{k+1} \right)$$

namely the chance that an A citizen by voting either makes a tie and wins the coin toss or breaks a tie where it would have lost the coin toss. Likewise, for the type B citizens

$$B_M^B = \sum_{k=0}^{\infty} \left(\frac{e^{-qN\alpha} (Nq\alpha)^k}{k!} \right) \left(\frac{e^{-(1-q)N\beta} ((1-q)N\beta)^k}{k!} \right) \frac{1}{2} \left(1 + \frac{qN\alpha}{k+1} \right)$$

Proposition 1. *There exists an equilibrium (α, β) in the M system. For uniqueness it suffices that F is weakly concave. The equilibrium has the following properties:*

⁶Recall that the interpretation is not restricted to electoral rules, as explained in the introduction. Two countries with the same electoral rule can have very different mappings from electoral outcomes to power shares, and this is the summary or reduced form variable that we are interested in and that affects turnout.

- Size effect:

$$\frac{dT_M}{dN} < 0$$

- Underdog effect with partial compensation:

$$q < 1/2 \implies \alpha > \beta, \quad q\alpha < (1 - q)\beta$$

as

$$(1) \quad q\alpha (F^{-1}(\alpha))^2 = (1 - q)\beta (F^{-1}(\beta))^2$$

To obtain the result we first used Myerson's approximation (see Myerson (1998)) to compute the value of the benefit side (B_M^A, B_M^B) . Equating the benefit side to the cost side we obtained a system of two equations in (α, β) , which we then show has a unique solution.

The size effect shows how the benefit of voting declines for larger electorates, although we will show that the rate of decline depends crucially on whether the parties do or do not have the same support ex-ante. The underdog effect shows that the party with less supporters has higher relative turnout. We discuss all these effects in the following section.

2.2. Discussion of Majority System. For any citizen the benefit of voting is proportional to the chance of being pivotal, i.e. the chance that the election outcome is a tie. The chance of a tied election is largest when the ex-ante chance that one of the two parties wins the election is 50%, namely when chance that any given voter is both an A supporter and votes is equal to the chance that he is both a B supporter and votes.

If the ex ante chance that any of the two parties wins is not 50%, then we have a leader party that is more likely to win the election and an underdog party. For any given strategy profile, if a supporter of the underdog party who is abstaining deviates and decides to vote instead, then the chance of a tie increases and hence the benefit of voting increases for all voters.

If all voters have the same cost of voting, then a strategy profile in which the ex-ante chance of one party winning is not 50% cannot be an equilibrium: if it were an equilibrium, then only the supporters of the underdog party who decide to vote should have a benefit of voting above the cost; however, a supporter of the underdog party who is abstaining, by deviating and going to vote would reap an even larger benefit as the chance of being pivotal for supporters of the underdog party increases with his additional vote. As a consequence, in any equilibrium with homogenous costs the chance that any party wins is 50%. In other words, the voters of the party with less supporters turn

out relatively more than the voters of the party with more supporters enough to have full compensation of the ex ante given preference split q .

The above argument breaks down and the 50-50 election outcome is in general not an equilibrium when the cost of voting is not homogenous but voting costs are extracted from the same distribution for all citizens regardless of party preferences. The party with less supporters which has to turn out more voters should have a higher cost threshold for voting, but in equilibrium this is not possible given that the benefit is the same.

Assume for instance that $q = 1/3$ so that one party has double the ex-ante support than the other. For a 50-50 election, i.e. in which we have $q\alpha = (1 - q)\beta$, the underdog party must turn out twice as much as the leader party, for instance we could have $(\alpha = 1/2, \beta = 1/4)$, and this cannot happen unless all citizens have the same cost. As a result, in equilibrium we will not have a 50-50 outcome, the underdog party supporters turn out relatively more but the ex-ante leader party is still more likely to win the election, namely we have only partial compensation. In formulas we have

$$q\alpha (F^{-1}(\alpha))^2 = (1 - q)\beta (F^{-1}(\beta))^2$$

so $q < 1/2$ implies $\alpha > \beta$ and $q\alpha < (1 - q)\beta$.

Homogenous cost would mean $F^{-1}(\alpha) = c = F^{-1}(\beta)$, which would imply full compensation and 50% ex ante chance of victory.

$$q\alpha = (1 - q)\beta$$

Of course when $q = 1/2$ then we have a 50-50 outcome with both homogenous and heterogenous costs.

Krasa & Polborn (2008) get a different result due to the fact that the cost distribution is bounded away from zero, so for a large enough population only the voters with voting costs approaching the lower bound vote. Therefore asymptotically their model becomes similar to a homogenous cost model with cost equal to the lower bound.

The distinction between homogenous and heterogeneous costs is very important and should be kept in mind if we want to compare turnout across different power sharing systems. The different equilibria with different cost assumptions, namely a 50-50 outcome versus a non 50-50

outcome, imply very different overall turnout numbers in large elections. In fact, the benefit of voting and hence the turnout are proportional to

$$B_M \sim \frac{e^{-N(\sqrt{q\alpha} - \sqrt{(1-q)\beta})^2}}{\sqrt{N}}$$

In the homogenous cost case, in which $q\alpha = (1-q)\beta$, this implies that turnout declines at the rate $N^{-1/2}$.

In the heterogeneous cost case, where $q\alpha \neq (1-q)\beta$ unless $q = 1/2$, turnout declines at the exponential rate e^{-N} for $q < 1/2$ and declines at the rate $N^{-1/2}$ when $q = 1/2$.⁷

2.3. Proportional System ($\gamma = P$). In a P system the share of power is proportional to the vote share obtained in the election. So if (a, b) are the absolute numbers of votes for each party, the power of parties A and B would be respectively $(\frac{a}{a+b}, \frac{b}{a+b})$.⁸

The expected marginal benefit of voting B_P^A is the expected increase in the vote share for the preferred party induced by a single vote, namely

$$B_P^A = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \left(\left(\frac{e^{-qN\alpha}(qN\alpha)^a}{a!} \right) \left(\frac{e^{-(1-q)N\beta}((1-q)N\beta)^b}{b!} \right) \left(\frac{a+1}{a+b+1} - \frac{a}{a+b} \right) \right)$$

$$B_P^B = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \left(\left(\frac{e^{-qN\alpha}(qN\alpha)^a}{a!} \right) \left(\frac{e^{-(1-q)N\beta}((1-q)N\beta)^b}{b!} \right) \left(\frac{b+1}{a+b+1} - \frac{b}{a+b} \right) \right)$$

Lemma 2. *The marginal benefit of voting in the proportional system has the closed form*

⁷Chamberlain and Rothshild (1981) obtain a similar result on rates of convergence in a model in which two candidates receive votes as binomial random variables. They assume no abstention, so the number of votes can be seen as flips of identical coins with a certain bias q . They show that if you toss an even number n of coins, the chance of obtaining the same number of heads and tails (the chance of a tie) drops asymptotically like $n^{-1/2}$ when the coins are unbiased ($q = 1/2$) and exponentially if the coins are biased ($q < 1/2$).

⁸We assume that if nobody votes, power is shared equally, namely

$$\frac{a}{a+b} = \frac{b}{a+b} = \frac{1}{2} \text{ for } a = b = 0$$

$$B_P^A = \frac{(1-q)\beta}{NT^2} - e^{-NT} \left(\frac{((1-q)\beta)^2 - (q\alpha)^2 + (1-q)\beta\frac{1}{N}}{2T^2} \right)$$

$$B_P^B = \frac{q\alpha}{NT^2} + e^{-NT} \left(\frac{((1-q)\beta)^2 - (q\alpha)^2 - q\alpha\frac{1}{N}}{2T^2} \right)$$

Using this lemma, the sum of the marginal benefits for the two types is

$$B_P^A + B_P^B = \frac{1}{NT} \left(1 - \frac{e^{-NT}}{2} \right)$$

Studying the asymptotic properties of this sum it is possible to obtain simplifications to the closed form expected benefit functions and obtain the following characterization results:

Proposition 3. *In the P system there is always a unique equilibrium (α, β) . The equilibrium has the following properties:*

- Size effect:

$$\frac{dT_P}{dN} < 0$$

- Underdog effect with partial compensation:

$$q < 1/2 \implies \alpha > \beta, \quad q\alpha < (1-q)\beta$$

as

$$(2) \quad q\alpha F^{-1}(\alpha) = (1-q)\beta F^{-1}(\beta)$$

2.4. Comparison. The size effect and the underdog effect although qualitatively similar are quantitatively different across the two institutional systems. We will return to the implications of the differences in terms of partial compensation later in the section, and we now concentrate on the main comparative result of the paper, namely the comparison of turnout incentives across systems.

Proposition 4. *Turnout is larger in a proportional system when there is a favorite party, while it is higher in a winner take all system if the election is expected to be very close. Namely, for N large we have:*

$$q \neq 1/2 \implies T_M < T_P$$

$$q = 1/2 \implies T_M > T_P$$

The intuition behind this result relies on how fast the marginal benefit of voting decreases in the two models as the electorate gets larger. The M system has two asymptotic regimes: it decreases exponentially for $q \neq 1/2$ and for $q = 1/2$ it decreases at the algebraic rate of $N^{-1/2}$.

Since we have only partial compensation from the underdog effect, then for any $q \neq 1/2$ the majority party is always the more likely side to win. Hence the chance of a tied election, which is what drives rational voters to turn out, is much smaller than in the case $q = 1/2$ for any population size N . The two rates of convergence derived above are not particular to the Poisson uncertainty of this model.⁹

The benefit from voting in the P system drops asymptotically at the intermediate rate of N^{-1} . This rate is independent of q as in the P system the chance of being the pivotal voter, i.e. the event of a tied election, has no special relevance.

It is perhaps now intuitive that a winner take all system, unlike a proportional one, should have two quite different rates of convergence regimes (although as we explained this is not the case with a degenerate cost distribution). Be that as it may, only an explicit computation could determine that the rate of convergence in the P system is quantitatively between the two rates of convergence in the M system: $N^{-1} \in (N^{-1/2}, e^{-N})$.

Regarding the underdog effect, we have already explained in section 2.2 that with heterogeneous costs full compensation is impossible in equilibrium unless $q = 1/2$, and the same explanation holds for the proportional system. However, it is worth remarking the following difference between the two systems in this matter:

Remark 5. *In both systems the ex-ante favorite party obtains the majority in a large election, but the underdog party has a higher turnout of its supporters. The underdog effects in the two models compare as follows*

$$\frac{\alpha_P F^{-1}(\alpha_P)}{\beta_P F^{-1}(\beta_P)} > \frac{\alpha_M F^{-1}(\alpha_M)}{\beta_M F^{-1}(\beta_M)}$$

In order to illustrate both the comparison in terms of turnout and the implications of the above remark, we now turn to a numerical example.

2.5. Example. Consider the cost distribution family

$$c \in [0, 1], \quad F(c) = c^{1/z}.$$

⁹Herrera & Martinelli (2006) analyze a majority rule election without population uncertainty. They introduce aggregate uncertainty in a different way, which allows to obtain a closed form for the chance of being pivotal, namely

$$\frac{(a+b)!}{2^{a+b+1} a! b!}$$

As it can be seen using Stirling's approximation, that marginal benefit for large a and b has exactly the square root decline on the diagonal ($a = b$) and the exponential decline off the diagonal ($a = \omega b$, $\omega \neq 1$).

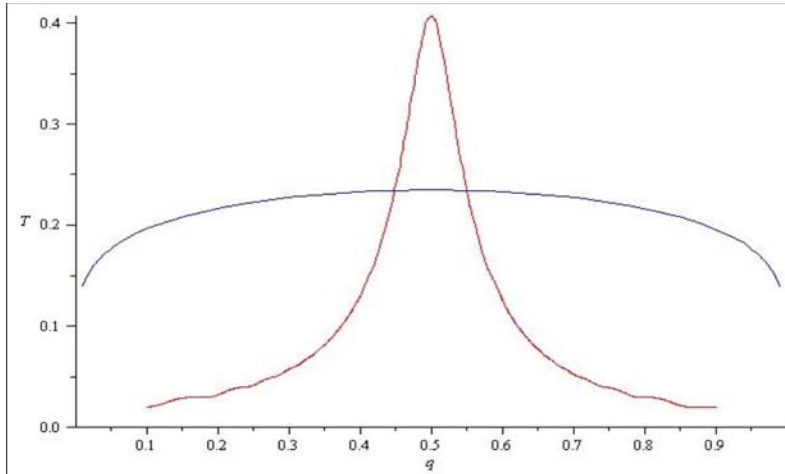
In this example the explicit solution for the proportional system yields

$$\begin{cases} \alpha_P = \left(\frac{1}{N} \frac{(1-q)q^{\frac{1}{z+1}}(1-q)^{\frac{1}{z+1}}}{\left(q(1-q)^{\frac{1}{z+1}} + (1-q)q^{\frac{1}{z+1}} \right)^2} \right)^{\frac{1}{z+1}} \\ \beta_P = \left(\frac{1}{N} \frac{q}{1-q} \frac{(1-q)q^{\frac{1}{z+1}}(1-q)^{\frac{1}{z+1}}}{\left(q(1-q)^{\frac{1}{z+1}} + (1-q)q^{\frac{1}{z+1}} \right)^2} \right)^{\frac{1}{z+1}} \end{cases}$$

On the other hand, the M equilibrium system is

$$\begin{aligned} \beta_M &= \left(\frac{q}{1-q} \right)^{\frac{1}{2z+1}} \alpha_M \\ \alpha_M^z &= \frac{e^{-N(\sqrt{(1-q)\beta_M} - \sqrt{q\alpha_M})^2}}{\sqrt{N}} \left(\frac{\sqrt{q\alpha_M} + \sqrt{(1-q)\beta_M}}{4\sqrt{\pi}(q(1-q)\alpha_M\beta_M)^{1/4}} \right) \end{aligned}$$

Setting $N = 3000$ and $z = 5$, the numerical solutions to the above systems yield a clear illustration of the comparative result of proposition 4. Below we compare as the preference split q varies the turnout T in the M and P systems: T_P is the flatter curve, while T_M shows the spike at $q = 1/2$.



The magnitudes of the turnout for each party (α, β) in the M and the P systems depends on the closeness of the election too. When the party B has the ex-ante advantage over party A, e.g. when $q = 1/3$, we have

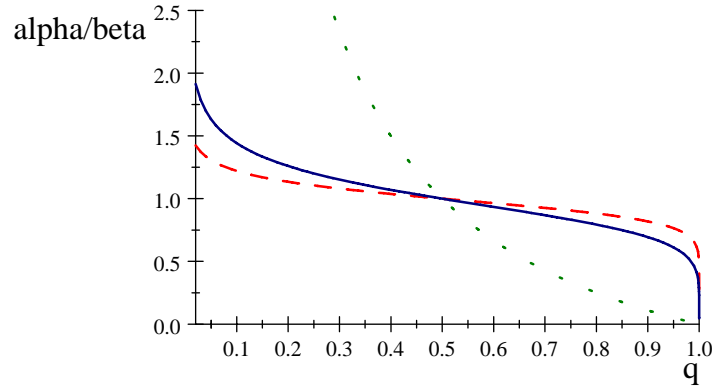
$q = 1/3$	α	β	T
P	24.8%	22%	23%
M	7.1%	6.7%	6.8%

Note in both the M and the P systems the presence of the ‘underdog effect’ ($\alpha > \beta$), and of ‘partial compensation’ ($q\alpha < (1 - q)\beta$).

When the election is close and no party has an ex ante advantage ($q = 1/2$), turnout in the majority system surpasses the turnout in the proportional system

$q = 1/2$	α	β	T
P	23.5%	23.5%	23.5%
M	40.9%	40.9%	40.9%

To compare the underdog effects, the following picture illustrates how the ratio α/β varies with q in the P system (continuous line) and in the M system (dashed line), and contrasts these decreasing curves with the steeper one that is obtained in the M system under homogeneous cost (dotted line) when there is ‘full compensation’ and the election is expected to be tied regardless of the initial preference split.



Applying Remark 5 to the numerical example, we have

$$\frac{\alpha_P F^{-1}(\alpha_P)}{\beta_P F^{-1}(\beta_P)} > \frac{\alpha_M F^{-1}(\alpha_M)}{\beta_M F^{-1}(\beta_M)} \implies \frac{\alpha_P^{z+1}}{\beta_P^{z+1}} > \frac{\alpha_M^{z+1}}{\beta_M^{z+1}} \implies \frac{\alpha_P}{\beta_P} > \frac{\alpha_M}{\beta_M}$$

so the underdog effect, by this measure, is larger in the P system than in the M system.

Remark 6. *In sum, there exist distributions of voting costs such that the underdog effect is higher in a proportional system. On the other hand, the size effect is higher in a proportional system only when the distribution of party supporters is close enough to symmetric.*

2.6. Extension to many parties. In this section we explicitly compute the equilibrium in the proportional system with three parties, and for more than three parties the derivations are analogous. The goal is to show that even when a proportional system allows for many parties, like suggested by Duverger hypothesis type arguments, the comparative result in terms of turnout is qualitatively analogous to the one obtained above. Moreover, we obtain a simple comparative statics result within the proportional system, namely that turnout increases in the number of parties.

Define

$$A := \alpha q_A N, \quad B := \beta q_B N, \quad C := \gamma q_C N$$

$$\text{with:} \quad q_A + q_B + q_C = 1$$

The marginal benefit for party A is: $B_P^A =$

$$\sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} \left(\frac{e^{-A} A^a}{a!} \right) \left(\frac{e^{-B} B^b}{b!} \right) \left(\frac{e^{-C} C^c}{c!} \right) \left(\frac{a+1}{a+b+c+1} - \frac{a}{a+b+c} \right)^{10}$$

Lemma 7. *The marginal benefit has the closed form*

$$B_P^A = \left(1 - \frac{A}{A+B+C} \right) \frac{1 - e^{-(A+B+C)}}{A+B+C} + \left(\frac{A}{A+B+C} - \frac{1}{3} \right) e^{-(A+B+C)}$$

The analog for the parties B or C is straightforward, and we obtain the following result:

Proposition 8. *(I) The comparison between turnout in the P system and the M system continues to hold even when there are multiple parties in the P system. (II) If parties are symmetric, turnout in the P system increases as the number of parties increases.*

¹⁰Assume again that if nobody votes, power is shared equally, namely

$$\frac{a}{a+b+c} = 1/3 \quad \text{for } a = b = c = 0$$

The fact that turnout increases with more parties is consistent with the observation that smaller parties obtain a higher turnout. The intuition for this follows from the following two observations. First, fixing the number of votes for the other parties z , the vote share increase for party A is

$$\left(\frac{a+1}{a+z+1} - \frac{a}{a+z} \right) = \left(\frac{a^2+a}{z} + 2a+1+z \right)^{-1}$$

which is larger for smaller values of the random variable a . Second, a smaller party (with a smaller q_A), assigns in the marginal benefit B_P^A a larger Poisson weight $\left(\frac{e^{-A}A^a}{a!} \right)$ to small values of a .

3. MOBILIZATION MODEL

The pivotal voter model features the well known free riding problem among voters, which makes turnout in a large election be typically small. Since in large elections the turnout is not always small, the free riding problem seems to be overcome in some way.¹¹ Regardless of how this might happen, it is important to know whether the turnout comparisons across electoral systems depend on the presence of this free riding problem, namely on whether the positive externality of voting is internalized or not. To address this question we introduce a voter mobilization model which will allow us to compare turnout across power sharing systems when parties' campaign efforts and spending are able to mobilize and coordinate citizens to go vote. Our goal is to see if we obtain different results from what we obtained in the pivotal voter model where the voting externality is not internalized.

In the rational voter model of participation studied so far, parties play no role. At the opposite extreme, mobilization models simplify the description of voters behavior and focus on parties strategies. In this section we aim to show that our comparative result is robust to the consideration of such mobilization efforts, even though the absolute properties of equilibrium in the two systems are different.

Assume that all voters have a benefit $G > 0$ from voting for their own preferred party, and voter v votes if and only if $G \geq c_v$. Since what matters is the net benefit, let's assume without loss of generality that the benefit G is constant across citizens and not affected by anything parties can do, whereas the distribution of heterogeneous costs can

¹¹Economists differ on how this collective action problem is by-passed in an election. For instance, Feddersen and Sandroni (2006) propose the idea that voters are ethical and receive a payoff from doing their duty and voting for their preferred candidate.

be affected by parties' spending.¹² Let us approximate here the large electorate with the unit interval, which is therefore also the support of the distribution of voting costs.

Let the distribution of voting costs among supporters of party i (known to both parties) be $F_i(c) = c^{\frac{1}{1+s_i}}$, with support \mathcal{R}_+ . The variable $s_i \geq 0$ represents the mobilization spending by party i .

Let s_A, s_B denote the effort/spending/campaigning level by the two parties (spending henceforth), to be determined in equilibrium of a simultaneous move game. Note that without any effort ($s_i = 0 \forall i$) the distribution is uniform. The spending costs are $l_i(s_i)$, increasing and convex, with $l_i(0) = l'_i(0) = 0$, $l'_i(s) > 0 \forall s > 0$, $\forall i$.

For any spending profile s , the vote share for party A is

$$(3) \quad x(s) = \frac{a(s)}{a(s) + b(s)} = \frac{qG^{\frac{1}{1+s_A}}}{qG^{\frac{1}{1+s_A}} + (1-q)G^{\frac{1}{1+s_B}}}$$

For each institutional setting γ and vote share x , party A has an expected power share $P_A^\gamma(x)$, and $P_B^\gamma(x) = 1 - P_A^\gamma(x)$. When choosing it's spending level, each party maximizes the utility function

$$U_i(s_i, s_{-i}) = P_i^\gamma(x(s_i, s_{-i})) - l_i(s_i).$$

The expected power share function that different parties may have in mind at the time of the spending decision depends on the institutional system: the closer the system is to pure winner-take-all, the steeper the increase of power share when going from a vote share slightly less than 1/2 towards the 1/2 threshold.¹³ On the other hand, the closer the system is to a consensus democracy the closer the power shares will be to be linear in the vote shares. Formally, we can capture this

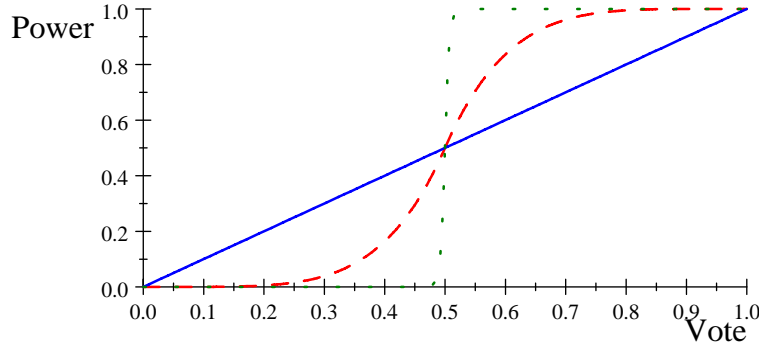
¹²As argued by Shachar and Nalebuff (1999), "parties decrease the direct cost of voting, for example they organize volunteers to drive people to the polls; they decrease the cost of becoming informed; they increase the cost of not voting by imposing social sanction on those who do not participate." Of course one could equivalently model mobilization efforts by parties as affecting benefits for given costs, saying that a party's spending makes all it's supporters feel the urgency of the moment, the intensity of the difference between having a ruler of one party or the other, as in Epstein, Morelli and O'Halloran (2009). These two approaches are obviously equivalent conceptually, but in this paper the assumption that mobilization efforts affect primarily the cost side of the equation is more convenient, for reasons that will be clear when both models will have been presented.

¹³In a winner-take-all system what matters is having the majority of votes, either because the majority of votes translates into obtaining all the seats in the Parliament, or because having the majority in the Parliament suffices to determine policies, without any concession to the minority party in the Parliament. Hence the expected power share is the probability of being the max party.

institutional determination of power sharing with a simple parameter $\gamma \geq 1$

$$P_A^\gamma(x) = \begin{cases} \frac{1}{2}(2x)^\gamma & \text{if } x < 1/2 \\ 1 - \frac{1}{2}(2(1-x))^\gamma & \text{if } x \geq 1/2 \end{cases}$$

Below we illustrate the power P_A^γ as a function of the vote share x for various power sharing parameters γ , namely $\gamma = 1$ i.e. the P system (continuous line), $\gamma = 5$ i.e. approaching the M system (dashed line), and $\gamma \rightarrow \infty$ i.e. a pure M system (dotted line).



Of course we have: $P_B^\gamma(x) = 1 - P_A^\gamma(x)$.¹⁴

$U_i(s_i, s_{-i})$ is continuous in s_i for every $\gamma \geq 1$. The best response $s_i^*(s_{-i})$ is certainly less than s_{-i} when s_{-i} goes to infinity; moreover, the best response to $s_{-i} = 0$ is strictly positive¹⁵ and hence an interior equilibrium must exist for every $\gamma \geq 1$.

Given $q \leq \frac{1}{2}$, assume that in equilibrium $x(s^*) \leq 1/2$, so that we can use just one of the two pieces of the power share function; then the validity of the assumption will be confirmed by the solution, since we prove that in equilibrium the two parties spend equal amounts in mobilization efforts.

Lemma 9. *The equilibrium spending level $s = s_A = s_B$ solves*

$$(4) \quad t'(s)(1+s)^2 = \frac{\gamma(1-q)(-\ln G)}{2}(2q)^\gamma.$$

¹⁴Note that if $\gamma = 1$ then power is linearly increasing in the vote share, whereas if $\gamma \rightarrow \infty$ the institutional system is winner take all.

¹⁵This is because it can be shown that the marginal utility of spending at 0,0 is

$$-\ln G * \gamma q(1-q)(2(1-q))^{\gamma-1} > 0.$$

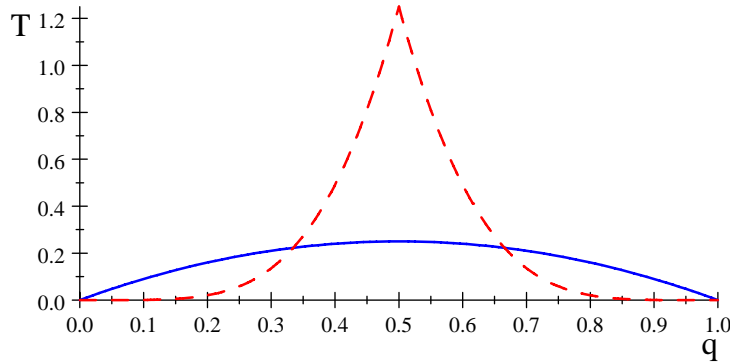
Having solved for the equilibrium spending level for every q and every $\gamma \geq 1$, it is possible to compute turnout, and we can conclude that

Proposition 10. (I) *There exists $\hat{q} \in (0, 1/2)$ such that for every $q \in (\hat{q}, 1/2)$ turnout is maximal for some intermediate $\gamma^*(q) > 1$;*
 (II) *When $q = 1/2$ turnout is strictly increasing in γ ; $\gamma^*(q)$ converges to infinity as q converges to $1/2$;*
 (III) *On the other hand, turnout is maximal with $\gamma = 1$ for every $q < \hat{q}$.*

Proposition 10 implies that if we compare turnout for $\gamma = 1$ (pure proportionality) and a high γ that approximates a winner take all system, the result depends crucially on how close the election is expected to be:

Corollary 11. *There exists $q^* \in (\hat{q}, 1/2)$ such that turnout is higher with a winner take all system than with a pure proportional system if and only if $q > q^*$.*

This result is also displayed in the picture below which represents party spending as a function of the closeness of the election q for $\gamma = 1$, i.e. the P system (continuous line), and for $\gamma = 5$, i.e. approximating the M system (dashed line).



We can see that this result is very similar to the main comparative result obtained in the rational voter model.

A few words about the choice of specific functional forms: (1) All the above results of the mobilization model are robust to changes in the specific functional form of the power function. For example, it is possible to check that if we used a power function similar to a contest success function typically employed in the contest literature (see e.g.

Hirschleifer (1989)), the qualitative results would be unchanged.¹⁶ (2) The choice of functional form for the distribution of costs of voting could also be changed to many others, but we chose this because it also works well for computational purposes in the rational voter model, as shown in example 1.

4. CONCLUDING REMARKS AND DIRECTIONS FOR FUTURE RESEARCH

In this paper we have shown that turnout of rational voters, for given distributions of partisan voters and voting costs, depends on the degree of proportionality of influence in the institutional system in the same way as when the turnout is mostly determined by mobilization efforts by parties. In both models, we have shown that turnout is higher in a winner take all system if the initial distribution of partisan voters is symmetric, whereas a more proportional system induces higher turnout otherwise. We have been able to compare underdog effect and size effect for relevant parameter values, and all the comparative results extend to the case in which a proportional system induces the existence of many parties.

We emphasize that our results do not need to invoke any of the standard arguments made about proportional representation, like fairness and representation reasons for turning out.¹⁷ The interaction effect of proportionality of influence and closeness of elections can be explained purely on the basis of rational calculus.

Even though the number of parties is exogenous in the paper, the fact that the comparative results in terms of turnout *do not* depend on the number of parties under the P system is reassuring, and makes the (hard) extension to endogenous party formation perhaps unnecessary. In light of the robustness results on the number of parties, even the extension to a multistage game in which the parties play some kind of legislative bargaining game after the election is not likely to generate any significant difference in terms of our main comparative results.

One theoretical extension that instead we aim to pursue is the following: what happens if we combine the two models we have studied? To be specific, what happens if we assume that when parties choose

¹⁶We prefer our formulation because it is a mapping from the vote share to the power share, whereas the contest function is not.

¹⁷For example, Jackman (1987) argued that “minor parties find it difficult to get their candidates elected in highly disproportional systems as their supporters may feel that their votes will be wasted and as a result may be inclined to abstain. PR is a fairer system and where people feel less alienated and are thus more inclined to vote.” Our formal results do not need to invoke fairness or representation.

their mobilization strategies they expect voters to compute their benefit of voting rationally as a function of the first stage mobilization efforts, rather than assuming a fixed benefit of voting? Could the equal spending result of the mobilization model be robust to this extension, or should we expect a change in some direction? In other words, when voters and parties are all players in the game, are their strategies complements or substitutes in the determination of turnout, given that when they are studied in isolation they determine the same comparative result?

Another theoretical question for future research is about mixed systems. Even though the comparison between the two extreme institutional systems is the same in the models considered here, it is possible, for some distributions of partisan voters, that turnout is maximal for some intermediate degree of proportionality of influence. This is definitely the case in the mobilization model, as one can see from proposition 5, but it has not been technically feasible to verify this possibility in the rational voter model. For the results to determine precise testable predictions it would be nice to characterize turnout incentives in mixed systems, because if the mapping from the degree of proportionality of influence to turnout is non monotonic, then we have to separate the prediction for close elections from that for asymmetric elections, since the expected sign of the coefficient of the proportionality variable depends on the initial conditions.

Beside the intrinsic value of the theoretical results, the findings of this paper could be useful for future empirical as well as experimental research. For example, if one focuses on voting rules, the empirical evidence on turnout in national elections (see e.g. Powell (1980, 1986), Crewe (1981), Jackman (1987) and Jackman and Miller (1995), Blais and Carthy (1990) and Franklin (1996)) all conclude that, everything else being equal, turnout is lower in plurality and majority elections than under Proportional Representation.¹⁸ On the other hand, experimental evidence (see Schram and Sonnemans (1996)) display the opposite finding. We have shown that these seemingly inconsistent findings are instead perfectly reconcilable, since the experimental design employed symmetry in the number of supporters for different parties – the case in which indeed we have shown that we should expect higher

¹⁸The standard caveat is that cross sectional studies are not to be considered conclusive evidence, because of the small sample size and few data points, cultural and idiosyncratic characteristics that are difficult to control for, as emphasized in Acemoglu (2005).

turnout under a winner take all system. Future experimental investigations should employ different treatments, allowing for the possibility of asymmetric distributions of partisan supporters and varying the degree of power proportionality. Similarly, we believe that the empirical analysis should be extended beyond electoral rules, since there are many other institutional details that affect the degree of proportionality of power as a function of the allocations of seats determined by the vote shares and the electoral formula. Finally, even the prediction that turnout should increase in the number of parties could be tested experimentally as well as on the existing field data.

APPENDIX

Proof of Proposition 1. Myerson's approximation for N large gives the following indifference conditions for the types with cost at the threshold ($c_\alpha = F^{-1}(\alpha)$, $c_\beta = F^{-1}(\beta)$)

$$B_M^A \simeq \frac{e^{-N(h-g)^2}}{\sqrt{N}} \frac{g+h}{4\sqrt{\pi}\sqrt{hg}} \frac{1}{g} = F^{-1}(\alpha)$$

$$B_M^B \simeq \frac{e^{-N(h-g)^2}}{\sqrt{N}} \frac{g+h}{4\sqrt{\pi}\sqrt{hg}} \frac{1}{h} = F^{-1}(\beta)$$

where we defined

$$g := \sqrt{q\alpha}, \quad h := \sqrt{(1-q)\beta_M(\alpha)}$$

The above system yields

$$\sqrt{q\alpha}F^{-1}(\alpha) = \sqrt{(1-q)\beta_M(\alpha)}F^{-1}(\beta)$$

Since the function $\sqrt{\alpha}F^{-1}(\alpha)$ is increasing we can define the function

$$\beta := \beta_M(\alpha)$$

where $\beta_M : [0, 1] \rightarrow [0, 1]$ is an increasing and differentiable function with $\beta_M(0) = 0$. The system is reduced to a single equation

$$B_M^A(\alpha, \beta_M(\alpha)) = F^{-1}(\alpha),$$

We now show existence of a solution to the above equation by showing that the two continuous functions on either side must cross at least once.

Assume wlog $q < 1/2$. We have

$$\alpha \in (0, 1] \implies g < h$$

and for any fixed N , we have

$$\lim_{\alpha \rightarrow 0} \frac{e^{-N(h-g)^2}}{\sqrt{N}} \frac{g+h}{4\sqrt{\pi}\sqrt{hg}} \frac{1}{g} > \lim_{\alpha \rightarrow 0} \frac{e^{-N(h-g)^2}}{\sqrt{N}} \frac{2}{4\sqrt{\pi}\sqrt{hg}} = \infty$$

For $\alpha = 1$ we have $h > g = \sqrt{q}$, so for all N above a certain value we have

$$\frac{e^{-N(h-g)^2}}{\sqrt{N}} \left(\frac{g+h}{4\sqrt{\pi}\sqrt{gh}} \frac{1}{g} \right) < 1$$

which proves existence of a solution, because $F^{-1}(\alpha)$ is increasing and $F^{-1}(1) = 1$.

For uniqueness we need to show that the B_M^A is decreasing in α , namely that the following quantity is negative

$$\begin{aligned} & \frac{d}{dg} \left(\frac{e^{-N(h-g)^2}}{\sqrt{N}} \frac{g+h}{4\sqrt{\pi}g\sqrt{hb}} \right) = \\ & = \frac{e^{-N(h-g)^2}}{\sqrt{N}} \left(-2N(h-g) \frac{d(h-g)}{dh} \frac{g+h}{4\sqrt{\pi}g\sqrt{gh}} + \frac{d}{dh} \left(\frac{g+h}{4\sqrt{\pi}g\sqrt{gh}} \right) \right) \end{aligned}$$

For large N this derivative will be negative if and only if

$$\frac{d(h-g)}{da} = \frac{\sqrt{1-q}}{\sqrt{q}} \frac{d\beta'}{d\alpha'} - 1 > 0$$

where we defined

$$\begin{aligned} \beta' & : = \sqrt{\beta}, \quad \alpha' := \sqrt{\alpha} \\ G(\alpha') & : = \alpha' F^{-1}((\alpha')^2) = \sqrt{\alpha} F^{-1}(\alpha) \end{aligned}$$

we have

$$\left(\sqrt{1-q} \right) G(\beta') = (\sqrt{q}) G(\alpha') \quad \implies \quad \frac{\sqrt{1-q}}{\sqrt{q}} \frac{d\beta'}{d\alpha'} = \frac{G'(\alpha')}{G'(\beta')}$$

So we need G' to be increasing

$$G'(\alpha') = \frac{d}{d\alpha} (\sqrt{\alpha} F^{-1}(\alpha)) \frac{d\alpha}{d\alpha'} = 2 \frac{d}{d\alpha} (\alpha F^{-1}(\alpha))$$

so it suffices for $\alpha F^{-1}(\alpha)$ to be weakly convex, so it suffices to have $F(\alpha)$ weakly concave.

As for the size effect, note that the marginal benefit side B_P^A decreases with N for all α while the cost side remains unchanged. Hence by the implicit function theorem as we increase N we have lower α which implies lower β and in turn lower turnout, formally

$$\begin{aligned} 0 & = \frac{d(B_M^A - F^{-1})}{d\alpha} \frac{d\alpha}{dN} + \frac{d(B_M^A - F^{-1})}{dN} \\ \frac{d\alpha}{dN} & = - \frac{\frac{dB_M^A}{dN}}{\frac{d(B_M^A - F^{-1})}{d\alpha}} < 0 \quad \implies \quad \frac{d\beta}{dN} < 0 \quad \implies \quad \frac{dT_M}{dN} < 0 \end{aligned}$$

The underdog effect is immediate, as F^{-1} is increasing

$$\begin{aligned} q\alpha (F^{-1}(\alpha))^2 &= (1-q)\beta (F^{-1}(\beta))^2 \\ q < 1/2 &\iff \alpha > \beta, \quad q\alpha < (1-q)\beta \end{aligned}$$

□

Proof of Lemma 2. For given (α, β) call the expected number of voters for each party $R := qN\alpha$, $S := (1-q)N\beta$, we have

$$B_P^A = e^{-R-S} \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \binom{R^a}{a!} \binom{S^b}{b!} \left(\frac{a+1}{a+b+1} - \frac{a}{a+b} \right)$$

By differentiating and integrating the summands and inverting the series and integral operators we have

$$\begin{aligned} \sum_{b=0}^{\infty} \frac{S^b}{b!} \frac{a}{a+b} &= \frac{a}{S^a} \sum_{b=0}^{\infty} \int_0^S \frac{d}{dr} \left(\frac{1}{b!} \frac{r^{a+b}}{a+b} \right) dr = \frac{a}{S^a} \int_0^S \sum_{b=0}^{\infty} \left(\frac{1}{b!} r^{a+b-1} \right) dr \\ &= \begin{cases} \frac{a}{S^a} \int_0^S r^{a-1} e^r dr & \text{for } a \geq 1 \\ 1/2 & \text{for } a = 0 \end{cases} \end{aligned}$$

and

$$\sum_{b=0}^{\infty} \frac{S^b}{b!} \frac{a+1}{a+b+1} = \frac{a+1}{S^{a+1}} \int_0^S r^a e^r dr$$

By inverting the series and integral operators again in the series over a , we have

$$\begin{aligned} B_P^A &= e^{-R-S} \left(\sum_{a=0}^{\infty} \frac{R^a}{a!} \left(\frac{a+1}{S^{a+1}} \int_0^S r^a e^r dr \right) - \sum_{a=1}^{\infty} \frac{R^a}{a!} \left(\frac{a}{S^a} \int_0^S r^{a-1} e^r dr \right) - \frac{1}{2} \right) \\ &= e^{-R-S} \left(\int_0^S \left(\frac{1}{S} \left(\sum_{a=0}^{\infty} \frac{\left(\frac{R}{S}r\right)^a}{a!} + \sum_{a=1}^{\infty} \frac{\left(\frac{R}{S}r\right)^a}{(a-1)!} \right) - \frac{R}{S} \sum_{a=1}^{\infty} \frac{\left(\frac{R}{S}r\right)^{a-1}}{(a-1)!} \right) e^r dr - \frac{1}{2} \right) \\ &= e^{-R-S} \left(\frac{1}{S^2} \int_0^S e^{(1+\frac{R}{S})r} (S - RS + Rr) dr - \frac{1}{2} \right) \\ &= e^{-R-S} \left(\frac{1-R}{S} \left(\frac{e^{S+R} - 1}{1 + \frac{R}{S}} \right) + \frac{R}{S^2 \left(1 + \frac{R}{S}\right)^2} \int_0^{S+R} e^r r dr - \frac{1}{2} \right) \\ &= \frac{S}{(R+S)^2} - \frac{e^{-(R+S)} S^2 - R^2 + S}{(R+S)^2} \frac{1}{2} \end{aligned}$$

and by symmetry

$$B_P^B(R, S) = B_P^A(S, R)$$

□

Proof of Proposition 3. We want to show first that NT diverges as N diverges. For every $\alpha > 0$ and $\beta > 0$, as $N \rightarrow \infty$ both B_P^A and B_P^B tend to zero. Hence, the cost side of the equation shows that the only possible solution to the system ($B_P^A = F^{-1}(\alpha)$, $B_P^B = F^{-1}(\beta)$) as $N \rightarrow \infty$ is $(\alpha = 0, \beta = 0)$. Summing the two equations of the P system we have

$$\frac{1}{NT} \left(1 - \frac{e^{-NT}}{2} \right) = F^{-1}(\alpha) + F^{-1}(\beta)$$

Since the RHS goes to zero the LHS will too, which means that NT must go to infinity.

In sum for N large, since the exponential terms e^{-NT} vanish faster than the hyperbolic terms, the system approximates to

$$\frac{(1-q)\beta}{NT^2} = F^{-1}(\alpha), \quad \frac{q\alpha}{NT^2} = F^{-1}(\beta)$$

which yields

$$\begin{aligned} q\alpha F^{-1}(\alpha) &= (1-q)\beta F^{-1}(\beta) \\ q < 1/2 &\iff \alpha > \beta \end{aligned}$$

Since the function $\alpha F^{-1}(\alpha)$ is increasing we can define

$$\beta := \beta_P(\alpha)$$

where $\beta_P(\alpha) : [0, 1] \rightarrow [0, 1]$ is an increasing differentiable function with $\beta_P(0) = 0$. We now reduced the P system to one equation

$$B_P^A := \frac{(1-q)\beta_P(\alpha)}{NT^2} = F^{-1}(\alpha)$$

We now show has one and only one solution.

The cost side $F^{-1}(\alpha)$ is increasing from 0 to 1. Uniqueness comes from the fact that the benefit side decreases in α as its derivative is proportional to

$$\begin{aligned} \frac{\partial B_P^A}{\partial \alpha} &\propto [\beta'_P(\alpha)(q\alpha + (1-q)\beta_P(\alpha)) - 2\beta_P(\alpha)(q + (1-q)\beta'_P(\alpha))] \\ &= -[((1-q)\beta - q\alpha)\beta'_P(\alpha) + 2q\beta_P(\alpha)] < 0 \end{aligned}$$

as

$$\alpha > \beta \implies q\alpha < q\alpha \frac{F^{-1}(\alpha)}{F^{-1}(\beta)} = (1-q)\beta$$

Existence comes from the fact that for α approaching zero the benefit diverges as for any fixed N we have

$$\lim_{\alpha \rightarrow 0} \frac{1}{N} \frac{(1-q)\beta_P(\alpha)}{(q\alpha + (1-q)\beta_P(\alpha))^2} > \lim_{\alpha \rightarrow 0} \frac{1}{N} \frac{(1-q)\beta_P}{\alpha} = \infty$$

because

$$\lim_{\alpha \rightarrow 0} \frac{\beta_P}{\alpha} = \lim_{\alpha \rightarrow 0} \frac{q}{1-q} \frac{F^{-1}(\alpha)}{F^{-1}(\beta_P)} > \frac{q}{1-q} > 0$$

and for $\alpha = 1$ we have eventually (i.e. for all N above a certain value),

$$\frac{1}{N} \left(\frac{(1-q)\beta_P(1)}{(q + (1-q)\beta_P(1))^2} \right) < F^{-1}(1) = 1$$

Hence a unique solution $(\alpha_P, \beta_P(\alpha_P))$ exists for the equilibrium problem.

The proofs for the size effect and the underdog effect are analogous to the ones obtained in the M system. \square

Proof of Proposition 4. Assuming the cost side $F^{-1}(\alpha)$ is the same in the two systems, it suffices to show that the benefit sides of the equations determining the equilibrium α are ranked.

For any $q \neq 1/2$ we need to show that eventually (i.e. for any N above a given \bar{N}) we have

$$B_M^A(\alpha, \beta_M(\alpha)) < B_P^A(\alpha, \beta_P(\alpha)), \quad \text{for all } \alpha \in (0, 1]$$

namely

$$\frac{e^{-N(\sqrt{q\alpha} - \sqrt{(1-q)\beta_M})^2}}{\sqrt{N}} \left(\frac{\sqrt{q\alpha} + \sqrt{(1-q)\beta_M}}{4\sqrt{\pi}(q(1-q)\alpha\beta_M)^{1/4}} \right) \frac{1}{\sqrt{q\alpha}} < \frac{1}{N} \frac{(1-q)\beta_P}{(q\alpha + (1-q)\beta_P)^2}$$

Rearranging we have

$$e^{-N(\sqrt{q\alpha} - \sqrt{(1-q)\beta_M})^2} \sqrt{N} < \frac{(1-q)\beta_P}{(q\alpha + (1-q)\beta_P)^2} \left(\frac{\sqrt{q\alpha} + \sqrt{(1-q)\beta_M}}{4\sqrt{\pi}(q(1-q)\alpha\beta_M)^{1/4}} \frac{1}{\sqrt{q\alpha}} \right)^{-1}$$

which is satisfied as LHS above converges to zero, whereas the RHS is a positive constant for all $\alpha \in (0, 1]$ because

$$\begin{aligned} \alpha \in (0, 1] &\implies \beta_P \in (0, 1], \quad \beta_M \in (0, 1] \\ q \neq 1/2 &\implies \sqrt{q\alpha} \neq \sqrt{(1-q)\beta_M(\alpha)} \end{aligned}$$

Hence, for any eventually we have

$$q \neq 1/2 \implies \alpha_M < \alpha_P$$

The symmetry property $\beta(q) = \alpha(1 - q)$ (which holds in both the M and P systems) implies

$$q \neq 1/2 \implies \beta_M < \beta_P$$

hence

$$q \neq 1/2 \implies T_M < T_P$$

For $q = 1/2$ we have $\alpha = \beta$ in both P and M systems. We need to show that eventually

$$B_M^A > B_P^A, \quad \alpha \in (0, 1]$$

namely

$$\frac{1}{\sqrt{N}} \left(\frac{2\sqrt{q\alpha}}{4\sqrt{\pi}} \right) \frac{1}{q\alpha} > \frac{1}{N} \left(\frac{q\alpha}{2(2q\alpha)^2} \right)$$

Rearranging we have

$$\sqrt{N} \left(\frac{1}{2\sqrt{\pi}\sqrt{q\alpha}} \right) > \left(\frac{1}{8q\alpha} \right)$$

which is satisfied as the RHS is a positive constant and the LHS increases to infinity. Hence

$$q = 1/2 \implies \alpha_M > \alpha_P \implies T_M > T_P$$

□

Proof of Remark 5. Given that for the M system we have

$$q\alpha_M (F^{-1}(\alpha_M))^2 = (1 - q) \beta_M (F^{-1}(\beta_M))^2$$

and for the P system we have

$$q\alpha_P (F^{-1}(\alpha_P)) = (1 - q) \beta_P (F^{-1}(\beta_P))$$

then, we obtain the result

$$\frac{1 - q}{q} = \frac{\alpha_P (F^{-1}(\alpha_P))}{\beta_P (F^{-1}(\beta_P))} = \frac{\alpha_M (F^{-1}(\alpha_M))^2}{\beta_M (F^{-1}(\beta_M))^2} > \frac{\alpha_M F^{-1}(\alpha_M)}{\beta_M F^{-1}(\beta_M)}$$

because $\text{wlog } q < 1/2$ implies that

$$\alpha_M > \beta_M$$

and since F^{-1} is increasing we have

$$\frac{F^{-1}(\alpha_M)}{F^{-1}(\beta_M)} > 1$$

□

Proof of Lemma 7. Express the following series by differentiating and integrating the summands and inverting the series and integral operators

$$\begin{aligned}
\sum_{b=0}^{\infty} \frac{B^b}{b!} \frac{a}{a+b+c} &= \frac{a}{B^{a+c}} \sum_{b=0}^{\infty} \int_0^B \frac{d}{dr} \left(\frac{1}{b!} \frac{r^{a+b+c}}{a+b+c} \right) dr \\
&= \frac{a}{B^{a+c}} \int_0^B \sum_{b=0}^{\infty} \left(\frac{1}{b!} r^{a+b+c-1} \right) dr \\
&= \begin{cases} \frac{a}{B^{a+c}} \int_0^B r^{a+c-1} e^r dr & \text{for } a \geq 1 \\ 1/3 & \text{for } a = c = 0 \end{cases}
\end{aligned}$$

and likewise

$$\sum_{b=0}^{\infty} \frac{B^b}{b!} \frac{a+1}{a+b+1} = \frac{a+1}{B^{a+c+1}} \int_0^B r^{a+c} e^r dr$$

We compute the marginal benefit for party A by inverting the series and integral operators again over the series over a .

$$\begin{aligned}
B_P^A &= e^{-(A+B+C)} \left(\sum_{c=0}^{\infty} \frac{C^c}{c!} \left(\frac{\sum_{a=0}^{\infty} \frac{A^a}{a!} \left(\frac{a+1}{B^{a+c+1}} \int_0^B r^{a+c} e^r dr \right)}{-\sum_{a=1}^{\infty} \frac{A^a}{a!} \left(\frac{a}{B^{a+c}} \int_0^B r^{a+c-1} e^r dr \right)} \right) - \frac{1}{3} \right) \\
&= e^{-(A+B+C)} \left(\sum_{c=0}^{\infty} \frac{C^c}{c!} \left(\int_0^B \frac{r^c}{B^{c+1}} \left(\sum_{a=0}^{\infty} \frac{(Ar/B)^a}{(a-1)!} + \sum_{a=0}^{\infty} \frac{(Ar/B)^a}{a!} \right) e^r dr \right. \right. \\
&\quad \left. \left. - \int_0^B \frac{r^{c-1}}{B^c} \left(\sum_{a=1}^{\infty} \frac{(Ar/B)^a}{(a-1)!} \right) e^r dr \right) - \frac{1}{3} \right) \\
&= e^{-(A+B+C)} \left(\sum_{c=0}^{\infty} \frac{C^c}{c!} \left(\int_0^B \frac{r^c}{B^{c+1}} \left((Ar/B) e^{(Ar/B)} + e^{(Ar/B)} \right) e^r dr \right. \right. \\
&\quad \left. \left. - \int_0^B \frac{r^{c-1}}{B^c} \left((Ar/B) e^{(Ar/B)} \right) e^r dr \right) - \frac{1}{3} \right)
\end{aligned}$$

Inverting the series and integral operators again over the series over c .

$$\begin{aligned}
B_P^A &= e^{-(A+B+C)} \left(\int_0^B \left((Ar/B) e^{(Ar/B)} + e^{(Ar/B)} \right) \left(\sum_{c=0}^{\infty} \frac{C^c}{c!} \frac{r^c}{B^{c+1}} \right) e^r dr \right. \\
&\quad \left. - \int_0^B \left((Ar/B) e^{(Ar/B)} \right) \left(\sum_{c=0}^{\infty} \frac{C^c}{c!} \frac{r^{c-1}}{B^c} \right) e^r dr - \frac{1}{3} \right) \\
&= e^{-(A+B+C)} \left(\int_0^B \left((Ar/B) e^{(Ar/B)} + e^{(Ar/B)} \right) \frac{e^{rC/B}}{B} e^r dr \right. \\
&\quad \left. - \int_0^B \left((Ar/B) e^{(Ar/B)} \right) \frac{e^{rC/B}}{r} e^r dr - \frac{1}{3} \right) \\
&= e^{-(A+B+C)} \left(\int_0^B \left(\left(\frac{A}{B} r e^{\frac{A+B+C}{B} r} + e^{\frac{A+B+C}{B} r} \right) \frac{1}{B} - \frac{A}{B} e^{\frac{A+B+C}{B} r} \right) dr - \frac{1}{3} \right)
\end{aligned}$$

Computing the integral and simplifying, we have

$$\begin{aligned}
B_P^A &= e^{-(A+B+C)} \left(\left(\left(AB \left(\frac{1-e^{A+B+C}}{(A+B+C)^2} + \frac{e^{A+B+C}}{A+B+C} \right) + \left(B \frac{e^{A+B+C}-1}{A+B+C} \right) \right) \frac{1}{B} \right) - \frac{1}{3} \right) \\
&= e^{-(A+B+C)} \left(A \frac{1-e^{A+B+C}}{(A+B+C)^2} + \frac{e^{A+B+C}-1}{A+B+C} + \frac{A}{A+B+C} - \frac{1}{3} \right) \\
&= \frac{B+C}{(A+B+C)^2} (1-e^{-(A+B+C)}) + \left(\frac{A}{A+B+C} - \frac{1}{3} \right) e^{-(A+B+C)} \\
&= \left(1 - \frac{A}{A+B+C} \right) \frac{1-e^{-(A+B+C)}}{A+B+C} + \left(\frac{A}{A+B+C} - \frac{1}{3} \right) e^{-(A+B+C)}
\end{aligned}$$

□

Proof of Proposition 8. A similar calculation gives the analogous result for r parties:

$$B_P^A(r) = \left(\left(1 - \frac{A}{A+B+C+\dots+r} \right) \frac{1-e^{-(A+B+C+\dots+r)}}{A+B+C+\dots+r} + \left(\frac{A}{A+B+C+\dots+r} - \frac{1}{r} \right) e^{-(A+B+C+\dots+r)} \right)$$

For large enough N , B_P^A approximates to

$$\begin{aligned}
B_P^A &\simeq \left(1 - \frac{A}{A+B+C+\dots+r} \right) \frac{1}{A+B+C+\dots+r} \\
&= \left(\frac{\beta q_B + \gamma q_C + \dots}{(\alpha q_A + \beta q_B + \gamma q_C + \dots)^2} \right) \frac{1}{N}
\end{aligned}$$

so the benefit still decreases as N^{-1} , which implies a higher turnout than in M except in the case when the two parties in M have the same ex-ante support: $q = 1/2$.

For r parties with equal ex-ante support we have

$$q_A = q_B = q_C = \dots = q_r = 1/r \implies \alpha = \beta = \gamma = \dots$$

the first order condition for a party becomes

$$\left(1 - \frac{1}{r} \right) \frac{1 - e^{-\alpha_r N}}{\alpha_r N} \approx \left(1 - \frac{1}{r} \right) \frac{1}{\alpha_r N} = F^{-1}(\alpha_r)$$

so the turnout for that party α_r increases in r . Overall turnout increases too as in this symmetric case we have.

$$T_r = \alpha_r$$

□

Proof of Lemma 9. Recall that the vote share is

$$x(s) = 1 - \frac{(1-q)G^{\frac{1}{1+s_B}}}{qG^{\frac{1}{1+s_A}} + (1-q)G^{\frac{1}{1+s_B}}}$$

Assume wlog $x < 1/2$.

The first order condition for party A is

$$\begin{aligned} l'_A(s_A) &= \gamma(2x)^{\gamma-1} \frac{\partial x(s)}{\partial s_A} \\ &= \gamma(2x)^{\gamma-1} \left(-\frac{\ln G}{(1+s_A)^2} \right) \frac{\left(qG^{\frac{1}{1+s_A}} \right) \left((1-q)G^{\frac{1}{1+s_B}} \right)}{\left(qG^{\frac{1}{1+s_A}} + (1-q)G^{\frac{1}{1+s_B}} \right)^2} \end{aligned}$$

For B we have

$$\begin{aligned} \frac{\partial x(s)}{\partial s_B} &= \frac{\partial}{\partial s_B} \left(\frac{(1-q)G^{\frac{1}{1+s_B}}}{qG^{\frac{1}{1+s_A}} + (1-q)G^{\frac{1}{1+s_B}}} \right) \\ &= - \left(-\frac{\ln G}{(1+s_B)^2} \right) \frac{\left(qG^{\frac{1}{1+s_A}} \right) \left((1-q)G^{\frac{1}{1+s_B}} \right)}{\left(qG^{\frac{1}{1+s_A}} + (1-q)G^{\frac{1}{1+s_B}} \right)^2} \end{aligned}$$

The first order condition for party B is

$$\begin{aligned} l'_B(s_B) &= -\gamma(2x)^{\gamma-1} \frac{\partial x(s)}{\partial s_B} \\ &= \gamma(2x)^{\gamma-1} \left(-\frac{\ln G}{(1+s_B)^2} \right) \frac{\left(qG^{\frac{1}{1+s_A}} \right) \left((1-q)G^{\frac{1}{1+s_B}} \right)}{\left(qG^{\frac{1}{1+s_A}} + (1-q)G^{\frac{1}{1+s_B}} \right)^2} \end{aligned}$$

Taking the ratio of the two first order conditions we obtain

$$l'_A(s_A)(1+s_A)^2 = l'_B(s_B)(1+s_B)^2$$

Since the function $l'(s)(1+s)^2$ is increasing this implies $s_A = s_B$ and hence $x = q$.

The equal spending solution $s_A = s_B = s$ solves the implicit equation

$$s : l'(s)(1+s)^2 = \frac{\gamma(1-q)(-\ln G)}{2} (2q)^\gamma$$

□

Proof of Proposition 10. (II): When $q = 1/2$ the equilibrium condition is

$$s : l'(s)(1+s)^2 = \frac{\gamma(-\ln G)}{4}$$

which is increasing γ : approaching a winner take all system spending and turnout increase, because the marginal benefit of spending increases.

(I) and (III): When $q < 1/2$ we have

$$s : l'(s)(1+s)^2 = \frac{\gamma(1-q)(-\ln G)}{2}(2q)^\gamma$$

The LHS is increasing in the effort, the RHS is independent of the effort and increases in γ if and only if

$$(5) \quad \gamma < -\frac{1}{\ln\left(\frac{1}{2q}\right)}$$

which is satisfied for q close to $1/2$ and is violated for sufficiently low q . Consider \hat{q} that solves

$$-\ln\left(\frac{1}{2q}\right) = 1 \implies \hat{q} \simeq 19.4\%$$

then for every $q \leq \hat{q}$ condition (5) cannot be satisfied, and hence pure proportionality maximizes turnout. \square

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